3) INTEGRAL RELATIONS FOR A CONTROL VOLUME:

Motivation: 2 paths in analyzing fluid motion

1) Try to describe the detailed flow pattern at every point (x, y, z) in the field

2) Work with a finite region, make a balance of flow in versus flow out & determine the gross flow effects such as the force & torque on a body or the total energy exchange.

1: "differential" approach that will be developed in Chp.4

2: "control volume method", which is the subject of this chapter.

We will develop the concept of the control volume & find the rate of change of an arbitrary gross field property, a result called the Reynolds transport theorem.

Then, we will apply this theorem to mass, linear momentum, angular momentum & energy, and derive the four basic control volume relations of fluid mechanics.

A special case of frictionless, shaft-work-free momentum & energy: the Bernoulli equation

3.1) Basic Physical Laws of Fluid mechanics:

Fluid static applications: more like fun than work.

They require only the density of the fluid & knowledge of the position of the free surface.

Most flow problems require the analysis of an arbitrary state variable fluid motion defined by the geometry, the boundary conditions & the laws of mechanics.

3 basic approaches to the analysis of arbitrary flow problems:

1) Control volume, or large-scale analysis (Chp. 3)
2) Differential, or small-scale analysis (Chp. 4)
3) Experimental, or dimensional analysis (Chp. 5)

They are roughly equal in importance.

Control volume analysis is accurate for any flow distribution & often based on "one-dimensional" property values at the boundaries.

It always gives useful "engineering" estimates.

The differential approach can, in principle, be applied to any problem. Only a few problems such as straight pipe flow yield to exact analytical solutions!

However, the differential equations can be modeled numerically & computational fluid dynamics (CFD) can be used to give good estimates for almost any geometry!

The dimensional analysis applies to any problem, whether analytical, numerical or experimental.

It is particularly useful to reduce to cost of experimentation.

Differential analysis of fluid dynamics : Euler & d'Alembert in the late 18th century

Control volume analysis: Bernoulli in 1753

Dimensional analysis: Rayleigh & Buckingham at the end of 19th century

Ludwig Prandtl: the founder of modern fluid mechanics (early 1900s)

Systems versus Control Volumes:

All the laws of mechanics are written for a system, which is defined as an arbitrary quantity of mass of fixed identity.

Everything external to the system is the surroundings.

The system is seperated from its surroundings by its boundaries.

The laws of mechanics state what happens when there is interaction between the system & its surroundings.

i) Conservation of mass : $m_{syst} = \text{const.}$ $\quad \frac{dm}{dt} = 0$

It is so obvious in solid mechanics problems that we often forget about it. In fluid mechanics, we must pay a lot attention to it.

ii) If the surroundings exert a net force $\vec{F}$ on the system, Newton's 2nd law states

$$\vec{F} = m\vec{a} = m\frac{d\vec{V}}{dt} = \frac{d}{dt}(m\vec{V})$$

In fluid mechanics, we call it the linear momentum relation.

iii) If the surroundings exert a net momentum $\vec{M}$ about the cm of the system, there will be a rotation effect:

$$\vec{M} = \frac{d\vec{H}}{dt} \qquad \vec{H} = \Sigma(\vec{r} \times \vec{v})\,\delta m$$: the angular momentum about the system around its cm

We call it the angular momentum relation.

For an arbitrary mass & arbitrary moment, $\vec{H}$ is quite complicated. In elementary dynamics, we study only a rigid body rotating about a fixed x axis:

$$M_x = I_x \frac{d\omega_x}{dt}$$

ω_x : angular velocity

I_x : mass moment of inertia about the x axis

Unfortunately, fluid systems are not rigid & rarely reduce to such a simple relation.

iv) If heat δQ is added to the system or the work δW is done by the system, the 1st law of thermodynamics states

$$dE = \delta Q - \delta W \quad \Rightarrow \quad \frac{dE}{dt} = \dot{Q} - \dot{W}$$

2nd Law of Thermodynamics : $dS \geq \frac{\delta Q}{T}$

dS : entropy change

δQ : heat added

T : absolute temperature

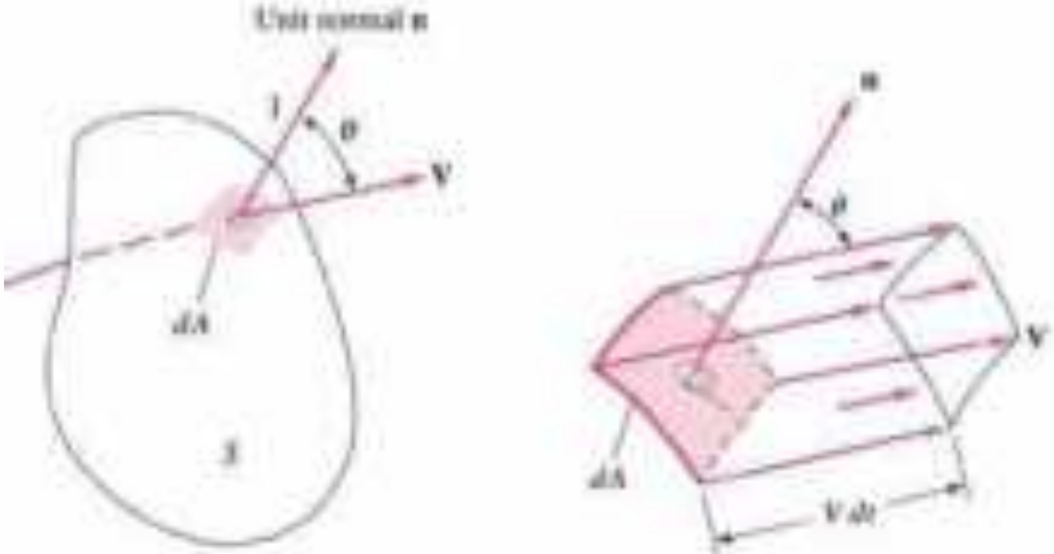
It is valid for any system & can also be written in control volume form. However, there are almost no practical applications except to analyze flow loss details.

Our aim in this chapter is to put four basic laws into the control volume form for arbitrary regions in a flow.

Whenever necessary, to complete the analysis, we will introduce a state relation such as the perfect gas law.

Volume & Mass Rate of the Flow:

All the analysis in this chapter involve the evaluation of a volume flow Q or mass flow $\dot{m}$ through a surface (imaginary) defined in the flow.

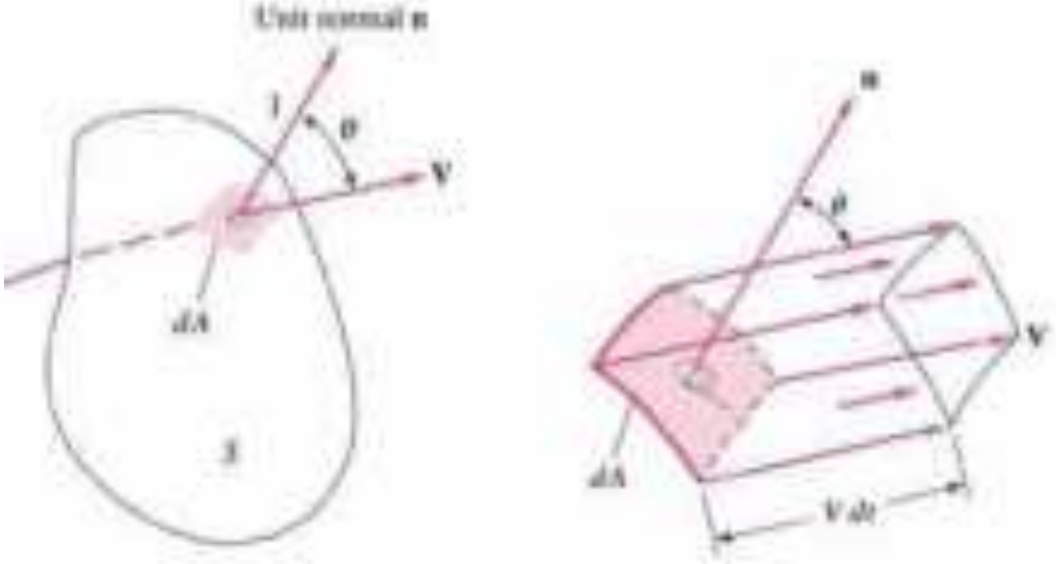


How much volume of fluid passes through S in unit time?

$$d\mathcal{V} = V\,dt\,dA\cos\theta$$

$$= \vec{V}\cdot\vec{n}\,dA\,dt$$

$\vec{n}$: outward unit normal to S



$Q \equiv$ integral of $\frac{d\upsilon}{dt}$ over the whole surface

= the total rate of flow through the surface S

$$\boxed{Q = \int_S (\vec{V}\cdot\vec{n})\, dA = \int_S V_n\, dA}$$

V_n: the component of $\vec{V}$ normal to dA.

Note that the dot product allows Q to have
a – sign for inflow and
a + sign for outflow.

$$dm = \rho \, d\mathcal{V} = \rho \, \vec{V} \cdot \vec{n} \, dA \, dt \qquad \dot{m} = \frac{dm}{dt}$$

Mass Flow :

$$\boxed{\dot{m} = \int_S \rho \, (\vec{V} \cdot \vec{n}) \, dA = \int_S \rho \, V_n \, dA}$$

If the density & velocity are constant over the surface S, we have the following one-dim. approximation:

$$Q = \int_S (\vec{V}\cdot\vec{n})\, dA = \int_S V_n\, dA = V_n \int dA = V_n A$$

When $V_n = V$, $Q = VA$

$$\dot{m} = \int_S \rho (\vec{V}\cdot\vec{n})\, dA = \rho \int_S (\vec{V}\cdot\vec{n})\, dA \Rightarrow \boxed{\begin{aligned}\dot{m} &= \rho Q \\ &= \rho V A\end{aligned}}$$

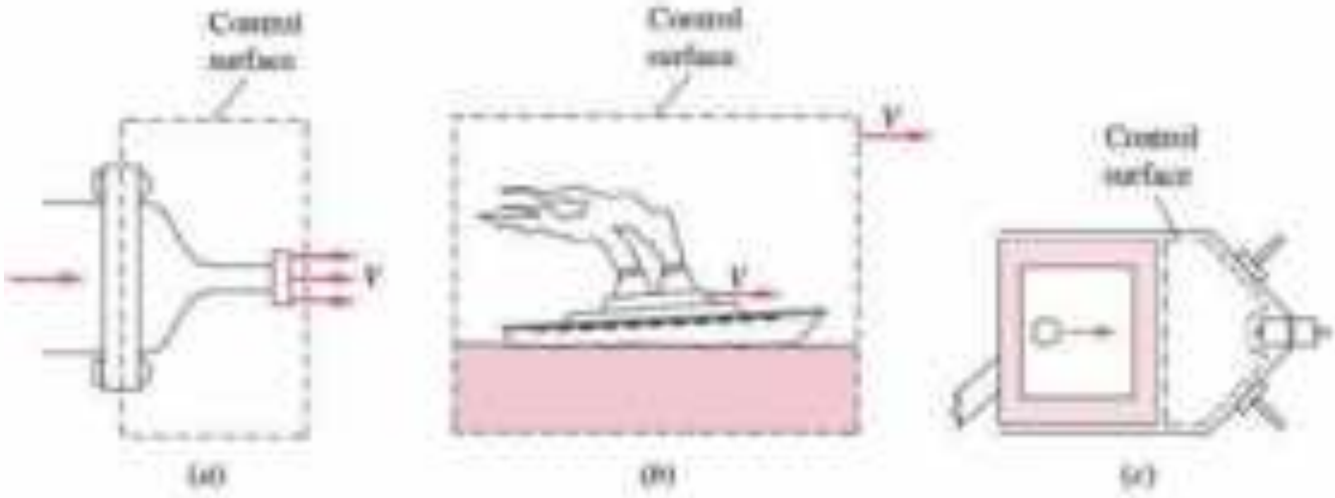
3.2) Reynolds Transport Theorem:

For the control volume analysis, we must convert our mathematics to apply to a specific region rather than to individual masses.

This conversion is called the Reynolds transport theorem & can be applied to all basic laws.

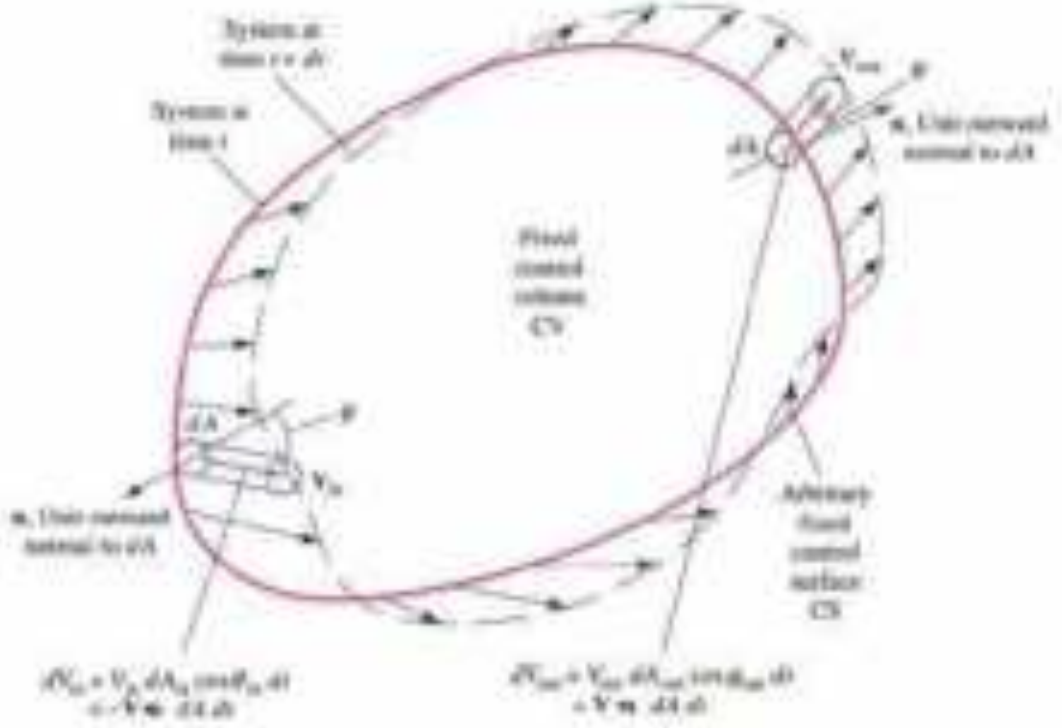
All the basic laws are concerned with time derivative of a system property to the rate of change of that property within a certain region.

Fig. 3.2 Fixed, moving, and deformable control volumes: (*a*) fixed control volume for nozzle stress analysis; (*b*) control volume moving at ship speed for drag force analysis; (*c*) control volume deforming within cylinder for transient pressure variation analysis.



The conversion formula differs slightly whether the control volume is fixed, moving or deformable.

Arbitrary Fixed Control volume:


System at time t + dt
System at time t
Fixed control volume CV
dA
V_{out}
n
n, Unit outward normal to dA
dA
n
V_{in}
n, Unit outward normal to dA
Arbitrary fixed control surface CS
$dV_{in} = V_{in}\, dA_{in} \cos\theta_{in}\, dt = -\mathbf{V}\cdot\mathbf{n}\, dA\, dt$
$dV_{out} = V_{out}\, dA_{out} \cos\theta_{out}\, dt = \mathbf{V}\cdot\mathbf{n}\, dA\, dt$

Some elemental areas will have inflow volume $(\vec{V}A\cos\theta)_{in}\,dt$ and others will have outflow volume $(\vec{V}A\cos\theta)_{in}\,dt$

Some surfaces might correspond to streamlines ($\theta = 90^\circ$) or solid walls ($\vec{V} = 0$)

B : any property of the fluid

$\beta \equiv \frac{dB}{dm}$; intensive value, amount of B per unit mass in any small element of the fluid

Bcv : total amount of B in the CV

$$B_{cv} = \int_{cv} dB = \int_{cv} \frac{dB}{dm}\, dm = \int_{cv} \beta\, dm = \int_{cv} \beta \rho\, d\forall$$

Examining the figure, we see three changes sources of changes of β in the CV

A change within the CV: $\frac{d}{dt}\int_{CV} \beta \rho \, d\forall$

Outflow of β from the CV: $\int_{CS} \beta \rho \, V \cos\theta \, dA_{out}$

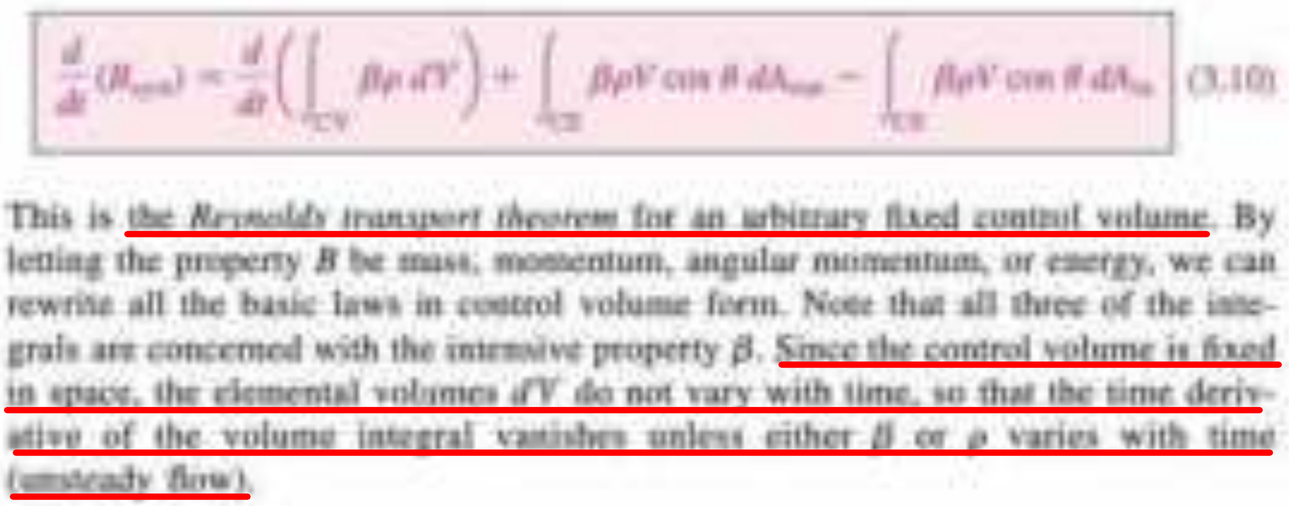
Inflow of β to the CV: $\int_{CS} \beta \rho \, V \cos\theta \, dA_{in}$

CV: Control volume CS: Control surface

$$\boxed{\frac{d}{dt} B_{sys} = \frac{d}{dt}\int_{CV} \beta \rho \, d\mathcal{V} + \int_{CS} \beta \rho V \cos\theta \, dA_{out} - \int_{CS} \beta \rho V \cos\theta \, dA_{in}}$$

This is the Reynolds transport theorem for an arbitrary fixed control volume.

By letting B = mass, momentum, angular momentum or energy, we can rewrite all the basic laws in control volume form.



$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta\rho \, d\mathcal{V}\right) + \int_{CS} \beta\rho V \cos\theta \, dA_{out} - \int_{CS} \beta\rho V \cos\theta \, dA_{in} \qquad (3.10)$$

This is *the Reynolds transport theorem for an arbitrary fixed control volume*. By letting the property B be mass, momentum, angular momentum, or energy, we can rewrite all the basic laws in control volume form. Note that all three of the integrals are concerned with the intensive property β. Since the control volume is fixed in space, the elemental volumes $d\mathcal{V}$ do not vary with time, so that the time derivative of the volume integral vanishes unless either β or ρ varies with time (unsteady flow).

B or β may be a vector or scalar property of the fluid.

Two alternative forms are possible for the flux terms.

Remember $\dot{m} = \int \rho V_n dA$ $\quad d\dot{m} = \rho V_n dA$: differential mass flux through the surface

Flux terms : $\int_{cs} \beta \rho V_n dA_{out} - \int_{cs} \beta \rho V_n dA_{in}$

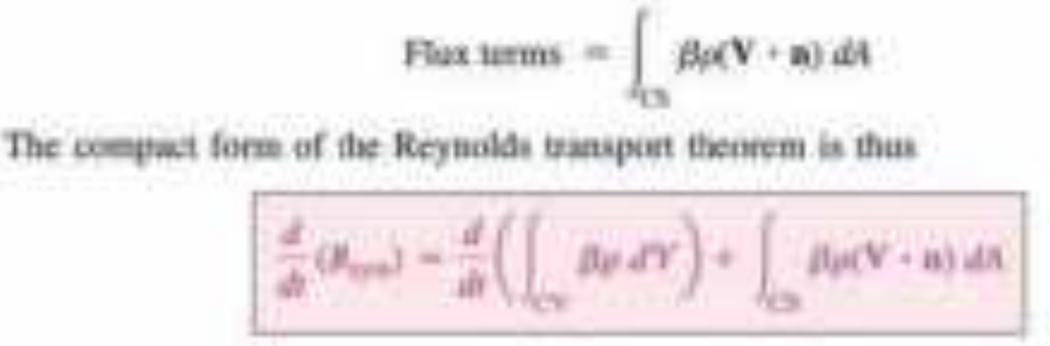
$$= \int_{cs} \beta \, d\dot{m}_{out} - \int_{cs} \beta \, d\dot{m}_{in}$$

This form helps us to visualize what is being calculated.

Alternative form: more elegant & compact

$\vec{n}$: outward normal vector everywhere on the CS

$\vec{V}\cdot\vec{n} = V_n$ for outflow $\vec{V}\cdot\vec{n} = -V_n$ for inflow



$$\text{Flux terms} = \int_{CS} \beta\rho(\mathbf{V}\cdot\mathbf{n})\,dA$$

The compact form of the Reynolds transport theorem is thus

$$\boxed{\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV}\beta\rho\,d\mathcal{V}\right) + \int_{CS}\beta\rho(\mathbf{V}\cdot\mathbf{n})\,dA}$$

The time derivative term can be written in the equivalent form

$$\frac{d}{dt}\left(\int_{CV} \beta\rho \, dV\right) = \int_{CV} \frac{\partial}{\partial t}(\beta\rho) \, dV$$

for the fixed control volume since the volume elements do not vary.

Steady flow: $\frac{\partial \beta}{\partial t} = 0$, $\frac{\partial \rho}{\partial t} = 0$

Therefore $\frac{d}{dt}\int_{cv} \beta\rho \, dV = 0$ for a steady flow in a fixed cv.

Control volume moving at Constant velocity:

If the control volume is moving uniformly at velocity $\vec{V}_s$, an observer fixed to the CV will see a relative velocity $\vec{V}_r$ of fluid crossing the CS, given by

$$\vec{V}_r = \vec{V} - \vec{V}_s$$

$\vec{V}$: fluid velocity wrt. the coordinate system in which the $\vec{V}_s$ is observed

The flux terms will be proportional to $\vec{V}_r$.

No change in the volume integral since CV moves as a fixed shape without deforming. Therefore, we have

$$\frac{d}{dt} B_{syst} = \frac{d}{dt}\int_{CV} \beta \rho \, d\forall + \int_{CS} \beta \rho \, (\vec{V}_r \cdot \vec{n}) \, dA$$

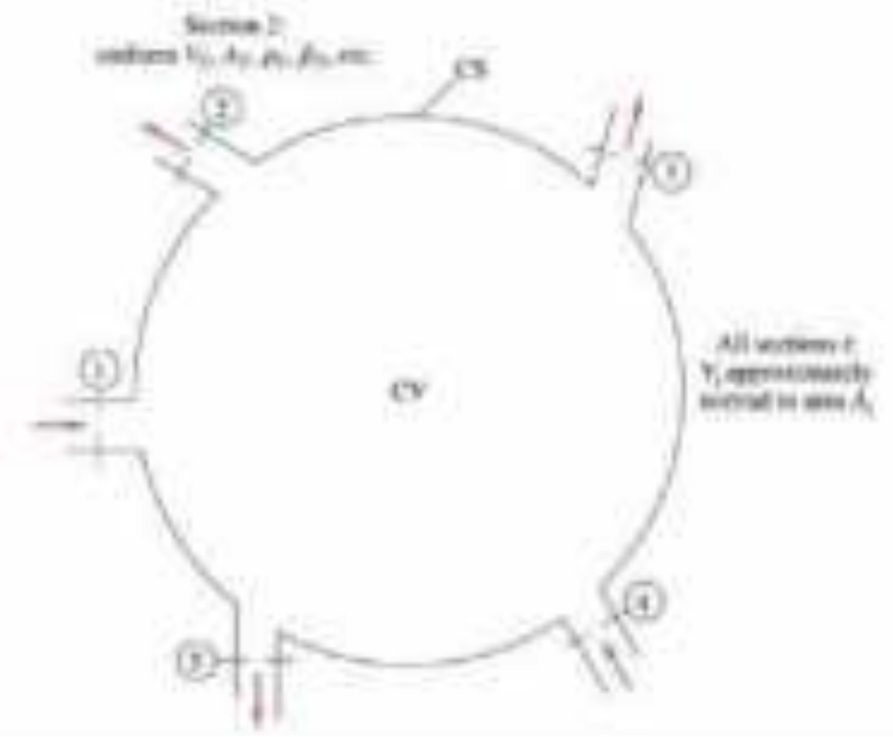
Note that it reduces to the previous form when $\vec{V}_s = 0$.

Control volume of Constant shape
but variable velocity

Arbitrarily moving & Deformable
Control volume

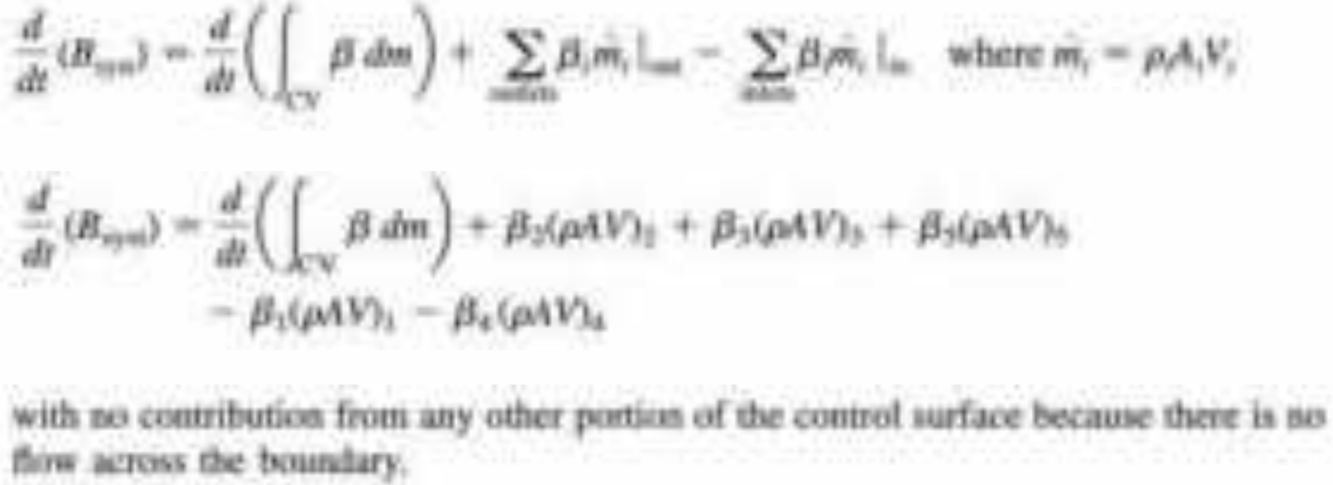
Omitted

One-dimensional Flux Term Approximations:



In many situations, the flow crosses the boundaries of the CS only at simplified inlets & exits that are approximately one-dimensional, i.e., the flow properties are nearly uniform over the cross-section.

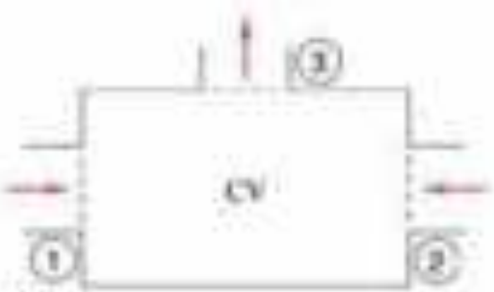
For a fixed CV, the surface integral reduces to a sum of positive (outlet) & negative (inlet) product terms for each cross section:



$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta \, dm\right) + \sum_{outlets} \beta_i \dot{m}_i \big|_{out} - \sum_{inlets} \beta_i \dot{m}_i \big|_{in} \quad \text{where } \dot{m}_i = \rho_i A_i V_i$$

$$\frac{d}{dt}(B_{syst}) = \frac{d}{dt}\left(\int_{CV} \beta \, dm\right) + \beta_2(\rho AV)_2 + \beta_3(\rho AV)_3 + \beta_5(\rho AV)_5 - \beta_1(\rho AV)_1 - \beta_4(\rho AV)_4$$

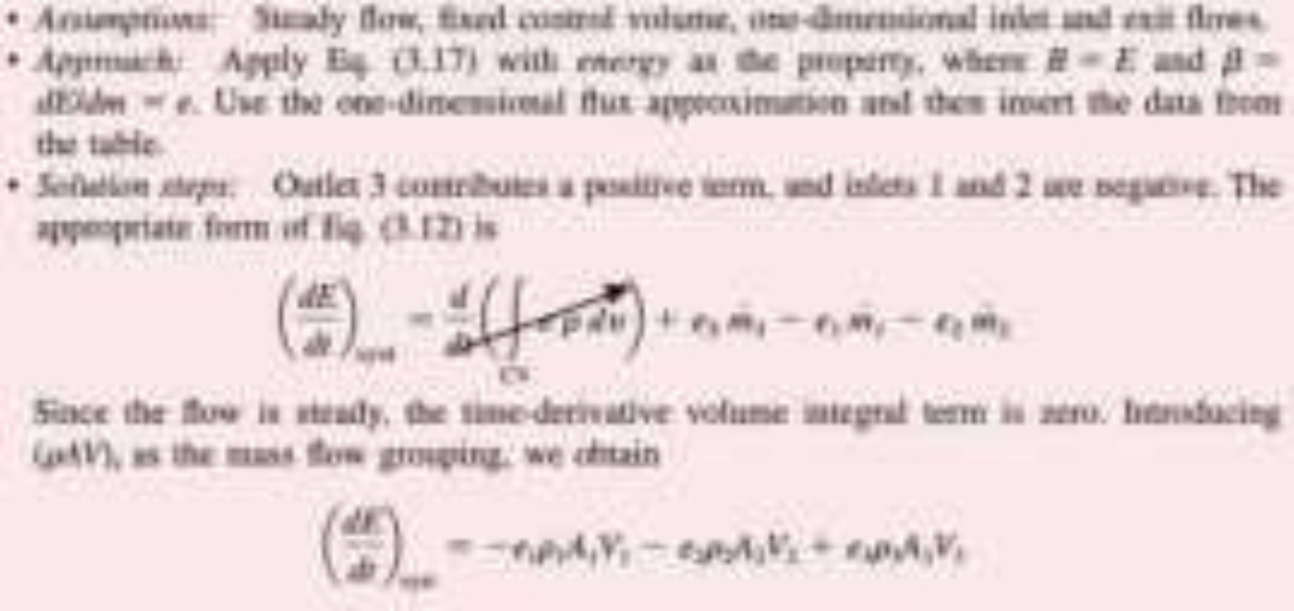
with no contribution from any other portion of the control surface because there is no flow across the boundary.



E3.1

A fixed control volume has three one-dimensional boundary sections, as shown in Fig. E3.1. The flow within the control volume is steady. The flow properties at each section are tabulated below. Find the rate of change of energy of the system that occupies the control volume at this instant.

Section	Type	ρ, kg/m^3	V, m/s	A, m^2	e, J/kg
1	Inlet	800	5.0	2.0	300
2	Inlet	800	8.0	3.0	100
3	Outlet	800	17.0	2.0	150



- *Assumptions:* Steady flow, fixed control volume, one-dimensional inlet and exit flows.
- *Approach:* Apply Eq. (3.17) with *energy* as the property, where $B = E$ and $\beta = dE/dm = e$. Use the one-dimensional flux approximation and then insert the data from the table.
- *Solution steps:* Outlet 3 contributes a positive term, and inlets 1 and 2 are negative. The appropriate form of Eq. (3.12) is

$$\left(\frac{dE}{dt}\right)_{syst} = \cancel{\frac{d}{dt}\left(\int_{CV} e\rho\, d\upsilon\right)} + e_3\,\dot{m}_3 - e_1\,\dot{m}_1 - e_2\,\dot{m}_2$$

Since the flow is steady, the time-derivative volume integral term is zero. Introducing $(\rho AV)_i$ as the mass flow grouping, we obtain

$$\left(\frac{dE}{dt}\right)_{syst} = -e_1\rho_1A_1V_1 - e_2\rho_2A_2V_2 + e_3\rho_3A_3V_3$$

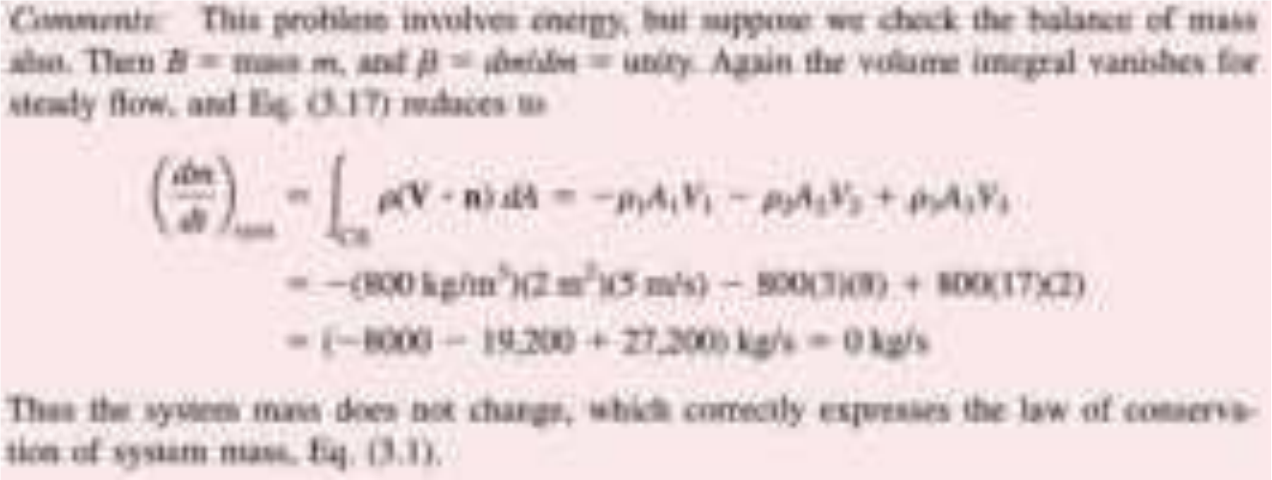
Introducing the numerical values from the table, we have

$$\left(\frac{dE}{dt}\right)_{\text{syst}} = -(300\text{ J/kg})(800\text{ kg/m}^3)(2\text{ m}^2)(5\text{ m/s}) - 100(800)(3)(8) + 150(800)(2)(17)$$

$$= (-2{,}400{,}000 - 1{,}920{,}000 + 4{,}080{,}000)\text{ J/s}$$

$$= -240{,}000\text{ J/s} = -0.24\text{ MJ/s} \qquad \text{Ans.}$$

Thus the system is losing energy at the rate of 0.24 MJ/s = 0.24 MW. Since we have accounted for all fluid energy crossing the boundary, we conclude from the first law that there must be heat loss through the control surface, or the system must be doing work on the environment through some device not shown. Notice that the use of SI units leads to a consistent result in joules per second without any conversion factors. We promised in Chap. 1 that this would be the case.



Comments: This problem involves energy, but suppose we check the balance of mass also. Then B = mass m, and $\beta = dm/dm$ = unity. Again the volume integral vanishes for steady flow, and Eq. (3.17) reduces to

$$\left(\frac{dm}{dt}\right)_{\text{syst}} = \int_{\text{CS}} \rho(\mathbf{V}\cdot\mathbf{n})\,dA = -\rho_1A_1V_1 - \rho_2A_2V_2 + \rho_3A_3V_3$$

$$= -(800\ \text{kg/m}^3)(2\ \text{m}^2)(5\ \text{m/s}) - 800(3)(8) + 800(17)(2)$$

$$= (-8000 - 19{,}200 + 27{,}200)\ \text{kg/s} = 0\ \text{kg/s}$$

Thus the system mass does not change, which correctly expresses the law of conservation of system mass, Eq. (3.1).

3.3) Conservation of mass

$$\frac{d}{dt} B_{syst} = \frac{d}{dt}\int_{CV} \beta \rho \, d\vartheta + \int_{CS} \beta \rho \, (\vec{V}_r \cdot \vec{n}) \, dA$$

where $\beta = \frac{dB}{dm}$. Take $B = m$: $\beta = 1$

$$\frac{dm_{syst}}{dt} = 0 = \frac{d}{dt}\int_{CV} \rho \, d\vartheta + \int_{CS} \rho \, (\vec{V}_r \cdot \vec{n}) \, dA$$

for a control volume moving at constant velocity.

For a fixed CV, we have

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \int_{CS} \rho (\vec{V} \cdot \vec{n}) dA = 0$$

If the CV has only a number of one-dim. inlets & outlets

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

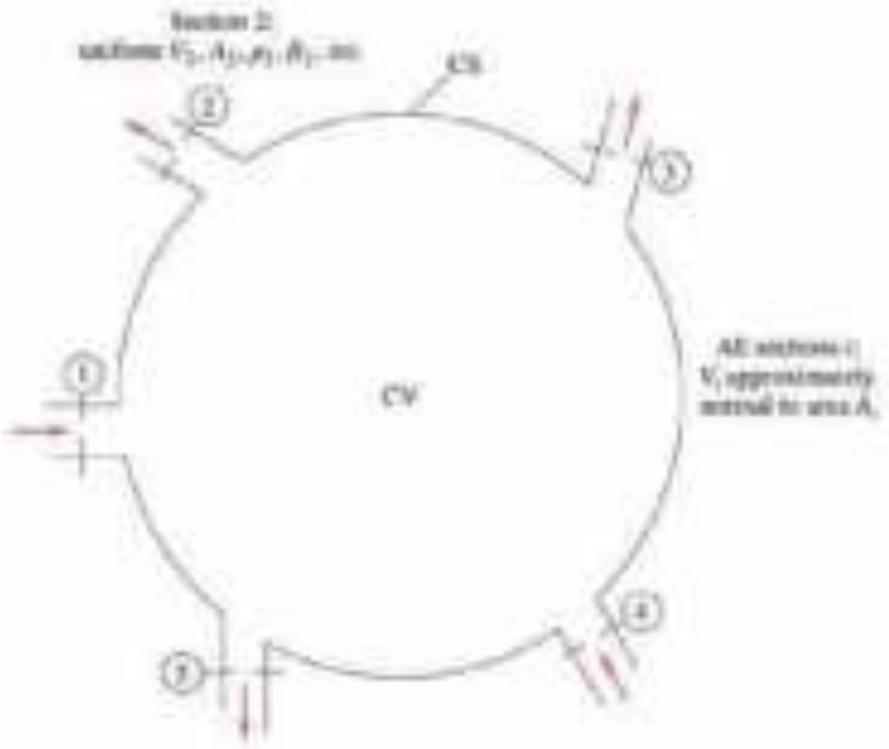
When the flow within the CV is steady, $\frac{\partial \rho}{\partial t} = 0$: $\int_{cs} \rho(\vec{V} \cdot \vec{n})\, dA = 0$

[we neglect sources & sinks that might be present in the CV.]

If the inlets & outlets are one-dimensional, we have for steady flow

$$\sum_i (\rho_i A_i V_i)_{in} = \sum_i (\rho_i A_i V_i)_{out}$$

This simple approximation is widely used in engineering analysis.



Outflow = Inflow

$$\rho_2 A_2 V_2 + \rho_3 A_3 V_3 + \rho_5 A_5 V_5$$

$$= \rho_1 A_1 V_1 + \rho_4 A_4 V_4$$

$\dot{m} = \rho A V$: mass flowing through the 1d cross section

$$\dot{m}_2 + \dot{m}_3 + \dot{m}_5 = \dot{m}_1 + \dot{m}_4$$

Steady flow mass conservation : $\sum_i (\dot{m}_i)_{out} = \sum_i (\dot{m}_i)_{in}$

$$\dot{m} = \rho A V$$

If the inlets & outlets are not 1d,

$$\dot{m}_{cs} = \int_{cs} \rho (\vec{V} \cdot \vec{n}) \, dA$$

Incompressible Flow: Further simplification is possible if the fluid is incompressible.

All liquids are nearly incompressible & gas flows can behave as if they were incompressible if the gas velocity is less than 30% of the speed of sound of the gas.

For a fixed CV & nearly incompressible flow, the term $\frac{\partial \rho}{\partial t}$ is small & the volume integral can be neglected.

$$\frac{d}{dt}\int_{CV}\frac{\partial \rho}{\partial t}\,d\vartheta + \int_{CS}\rho\,(\vec{V}\cdot\vec{n})\,dA = 0 = \rho\int_{CS}\vec{V}\cdot\vec{n}\,dA$$

$$\Rightarrow \int_{CS}\vec{V}\cdot\vec{n}\,dA = 0$$

If the inlets & outlets are 1d,

$$\sum_i (V_i A_i)_{out} = \sum_i (V_i A_i)_{in}$$

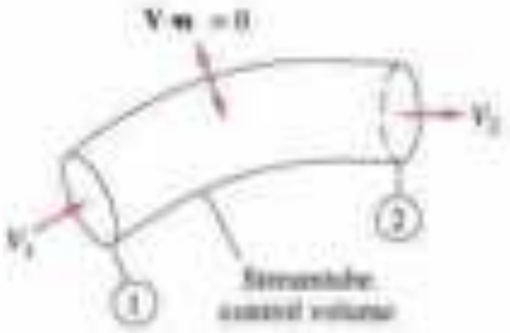
$$\sum Q_{out} = \sum Q_{in} \qquad Q_i = V_i A_i : \text{volume flow}$$

Average Velocity: $V_{av} = \frac{Q}{A} = \frac{1}{A} \int_{cs} \vec{V} \cdot \vec{n} \, dA$

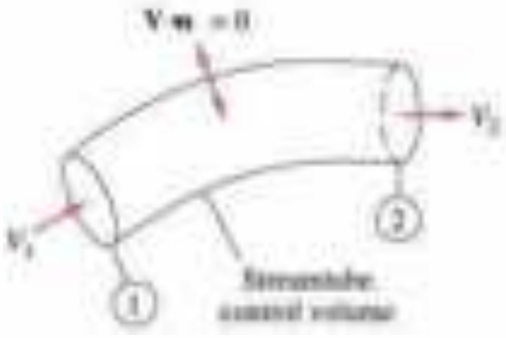
Average Density: $\rho_{av} = \frac{1}{A} \int_{cs} \rho \, dA$ (average over the cross section)

$$(\rho V)_{av} = \frac{1}{A} \int_{cs} \rho (\vec{V} \cdot \vec{n}) \, dA \approx \rho_{av} V_{av}$$

(we can often neglect the difference or, if necessary, use a correction factor between $\dot{m}$



Write the conservation-of-mass relation for steady flow through a streamtube (flow everywhere parallel to the walls) with a single one-dimensional inlet 1 and exit 2



Write the conservation-of-mass relation for steady flow through a streamtube (flow everywhere parallel to the walls) with a single one-dimensional inlet 1 and exit 2

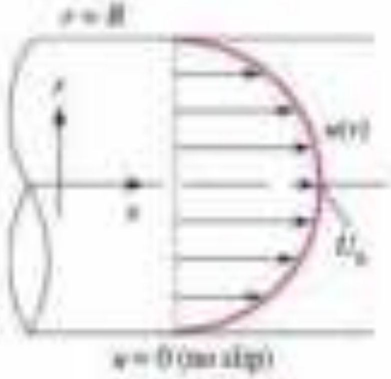
For steady flow Eq. (3.24) applies with the single inlet and exit:

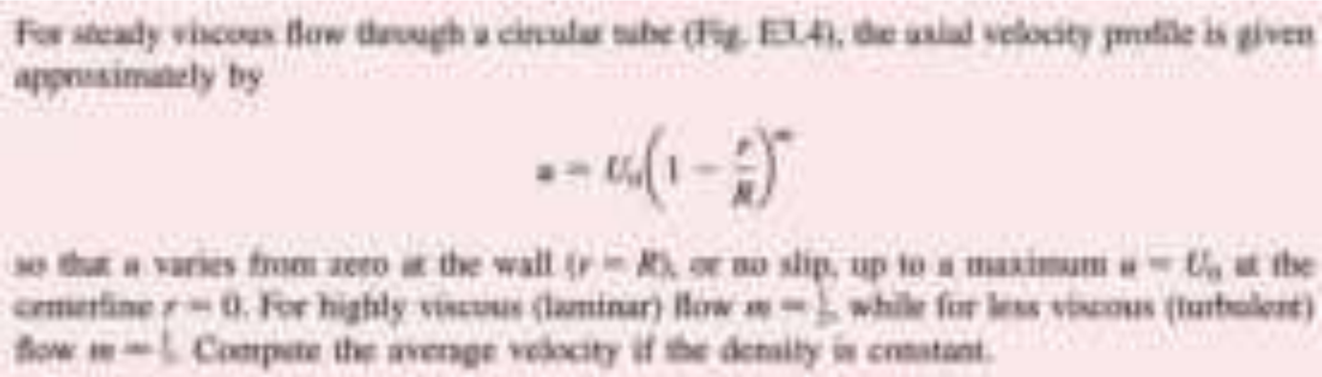
$$\dot{m} = \rho_1 A_1 V_1 = \rho_2 A_2 V_2 = \text{const}$$

Thus, in a streamtube in steady flow, the mass flow is constant across every section of the tube. If the density is constant, then

$$Q = A_1 V_1 = A_2 V_2 = \text{const} \qquad \text{or} \qquad V_2 = \frac{A_1}{A_2} V_1$$

The volume flow is constant in the tube in steady incompressible flow, and the velocity increases as the section area decreases. This relation was derived by Leonardo da Vinci in 1500.




For steady viscous flow through a circular tube (Fig. E3.4), the axial velocity profile is given approximately by

$$u = U_0\left(1 - \frac{r}{R}\right)^m$$

so that u varies from zero at the wall ($r = R$), or no slip, up to a maximum $u = U_0$ at the centerline $r = 0$. For highly viscous (laminar) flow $m = \frac{1}{2}$, while for less viscous (turbulent) flow $m = \frac{1}{7}$. Compute the average velocity if the density is constant.

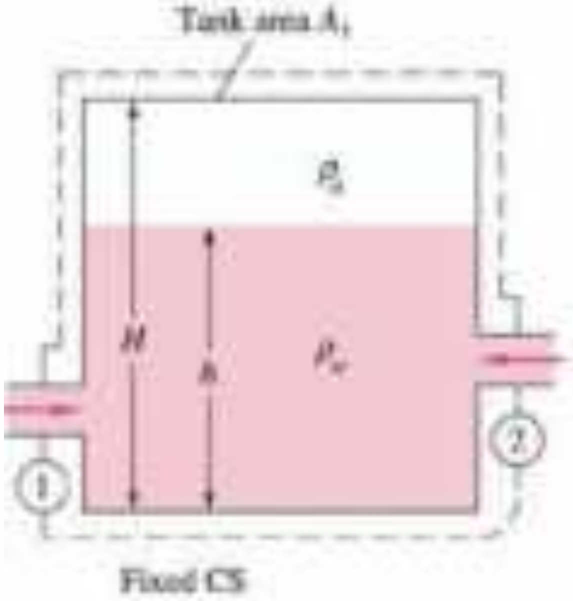
The average velocity is defined by Eq. (3.32). Here $\mathbf{V} = \mathbf{i}u$ and $\mathbf{n} = \mathbf{i}$, and thus $\mathbf{V} \cdot \mathbf{n} = u$. Since the flow is symmetric, the differential area can be taken as a circular strip $dA = 2\pi r\,dr$. Equation (3.32) becomes

$$V_{av} = \frac{1}{A}\int u\,dA = \frac{1}{\pi R^2}\int_0^R U_0\left(1 - \frac{r}{R}\right)^m 2\pi r\,dr$$

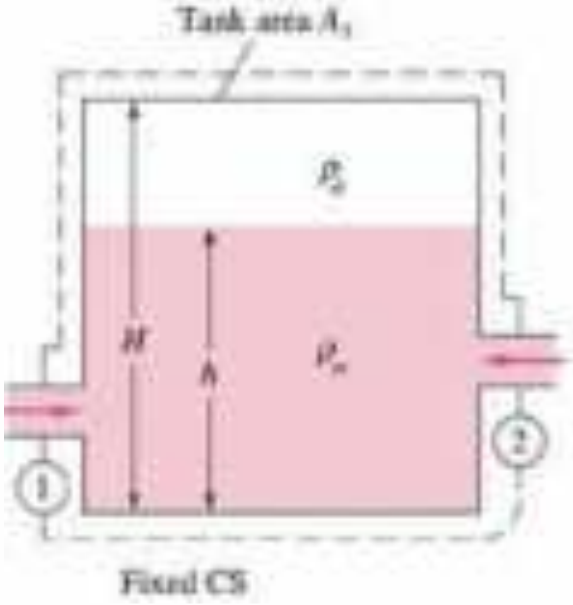
or

$$V_{av} = U_0\frac{2}{(1 + m)(2 + m)} \qquad \textit{Ans.}$$

For the laminar flow approximation, $m \approx \frac{1}{2}$ and $V_{av} \approx 0.53U_0$. (The exact laminar theory in Chap. 6 gives $V_{av} = 0.50U_0$.) For turbulent flow, $m \approx \frac{1}{7}$ and $V_{av} \approx 0.82U_0$. (There is no exact turbulent theory, and so we accept this approximation.) The turbulent velocity profile is more uniform across the section, and thus the average velocity is only slightly less than maximum.



The tank in Fig. E3.5 is being filled with water by two one-dimensional inlets. Air is trapped at the top of the tank. The water height is h. (*a*) Find an expression for the change in water height dh/dt. (*b*) Compute dh/dt if $D_1 = 1$ in, $D_2 = 3$ in, $V_1 = 3$ ft/s, $V_2 = 2$ ft/s, and $A_t = 2$ ft^2, assuming water at 20°C.



$$\frac{d}{dt}\int_{CV}\rho\, d\vartheta - \rho_1 A_1 V_1 - \rho_2 A_2 V_2 = 0 \quad \text{----} \; (*)$$

$$\int_{CV}\rho\, d\vartheta = \rho_w A_t h + \rho_a A_t (H-h)$$

$$\frac{d}{dt}\int_{CV}\rho\, d\vartheta = \rho_w A_t h + \underbrace{\frac{d}{dt}\left[\rho_a A_t (H-h)\right]}_{=\text{ rate of change of air mass}}$$

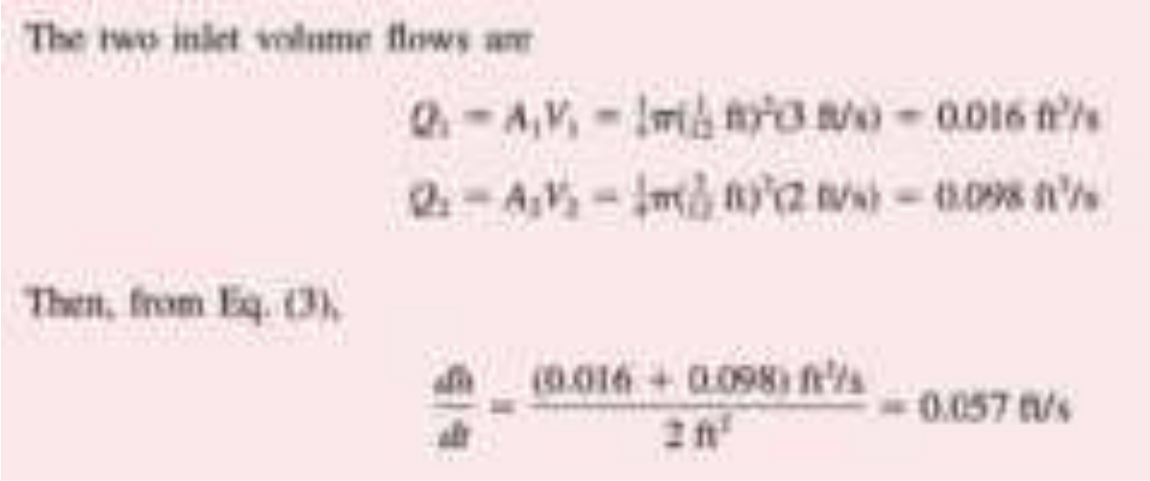
$= 0$ since the air is trapped.

$$\frac{d}{dt}\int_{CV}\rho\, d\vartheta = \rho_w A_t \frac{dh}{dt}$$: Insert into (*)

$$\frac{dh}{dt} = \frac{\rho_1 A_1 V_1 + \rho_2 A_2 V_2}{\rho_w A_t}$$

In this problem : $\rho_1 = \rho_2 = \rho_w$

$$\frac{dh}{dt} = \frac{A_1 V_1 + A_2 V_2}{A_t} = \frac{Q_1 + Q_2}{A_t}$$



The two inlet volume flows are

$$Q_1 = A_1V_1 = \tfrac{1}{4}\pi(\tfrac{1}{12}\text{ ft})^2(3\text{ ft/s}) = 0.016\text{ ft}^3/\text{s}$$

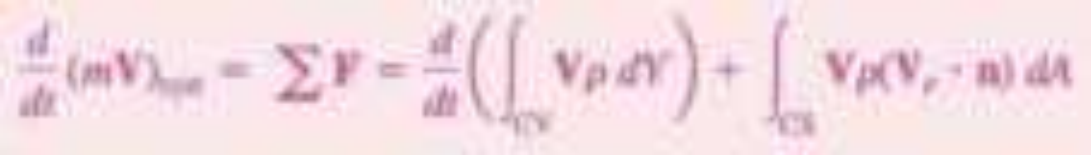
$$Q_2 = A_2V_2 = \tfrac{1}{4}\pi(\tfrac{3}{12}\text{ ft})^2(2\text{ ft/s}) = 0.098\text{ ft}^3/\text{s}$$

Then, from Eq. (3),

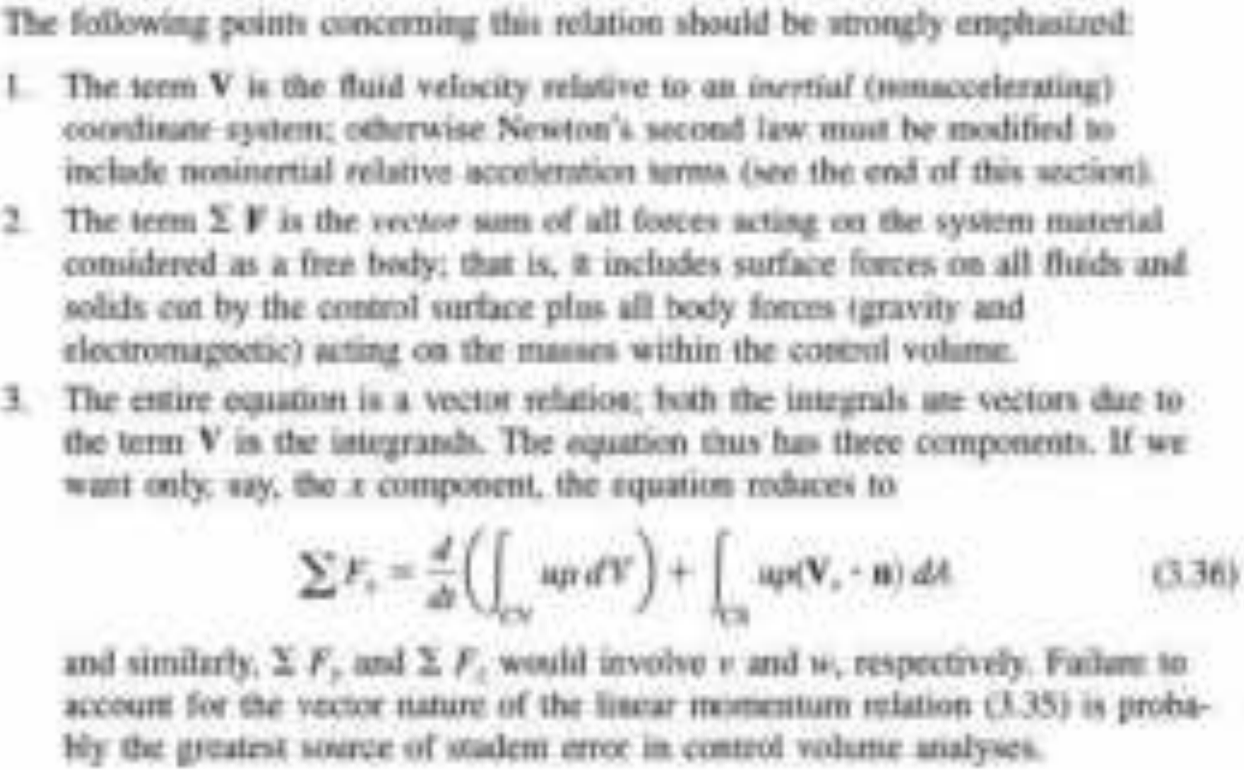
$$\frac{dh}{dt} = \frac{(0.016 + 0.098)\text{ ft}^3/\text{s}}{2\text{ ft}^2} = 0.057\text{ ft/s}$$

3.4) The Linear Momentum Equation:

$\vec{B} = m\vec{V} \quad \vec{\beta} = \frac{d\vec{B}}{dm} = \vec{V}$:



$$\frac{d}{dt}(m\mathbf{V})_{syst} = \sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA$$



The following points concerning this relation should be strongly emphasized:

1. The term **V** is the fluid velocity relative to an *inertial* (nonaccelerating) coordinate system; otherwise Newton's second law must be modified to include noninertial relative acceleration terms (see the end of this section).
2. The term Σ **F** is the *vector* sum of all forces acting on the system material considered as a free body; that is, it includes surface forces on all fluids and solids cut by the control surface plus all body forces (gravity and electromagnetic) acting on the masses within the control volume.
3. The entire equation is a vector relation; both the integrals are vectors due to the term **V** in the integrands. The equation thus has three components. If we want only, say, the x component, the equation reduces to

$$\sum F_x = \frac{d}{dt}\left(\int_{CV} u\rho\, d\mathcal{V}\right) + \int_{CS} u\rho(\mathbf{V}_r \cdot \mathbf{n})\, dA \qquad (3.36)$$

and similarly, ΣF_y and ΣF_z would involve v and w, respectively. Failure to account for the vector nature of the linear momentum relation (3.35) is probably the greatest source of student error in control volume analyses.

For a fixed control volume, the relative velocity $\mathbf{V}_r \equiv \mathbf{V}$, and Eq. (3.35) becomes

$$\boxed{\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho \, d\mathcal{V}\right) + \int_{CS} \mathbf{V}\rho(\mathbf{V} \cdot \mathbf{n}) \, dA} \qquad (3.37)$$

Again we stress that this is a vector relation and that **V** must be an inertial-frame velocity. Most of the momentum analyses in this text are concerned with Eq. (3.37).

One-dimensional Momentum Flux:

mass flux: $\dot{m} = \int_{cs} \rho (\vec{V} \cdot \vec{n}) dA$

Analogously, we can define the momentum flux:

$$\dot{\vec{M}}_{cs} = \int_{cs} \vec{V} \rho (\vec{V} \cdot \vec{n}) dA$$

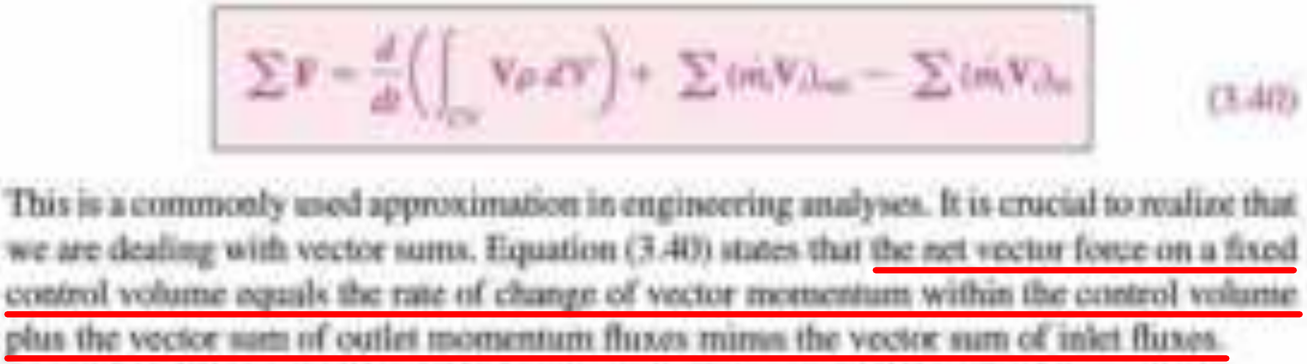
Dot product: $-$ for inlet momentum flux
$+$ for outlet momentum flux

When the cross section is 1d & $\vec{V}$ and ρ are uniform over the area:

$$\vec{\dot{M}}_i = \vec{V}_i \left(\rho_i V_{ni} A_i\right) = \dot{m}_i \vec{V}_i \quad \text{for outlet flux}$$

$$= -\dot{m}_i \vec{V}_i \quad \text{for inlet flux}$$

This is a commonly used approximation in engineering analysis.



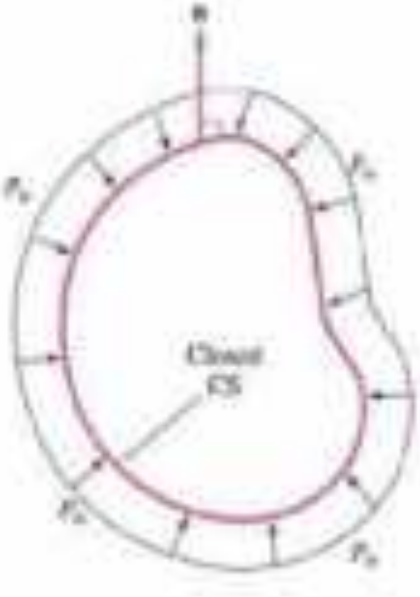
$$\boxed{\sum \mathbf{F} = \frac{d}{dt}\left(\int_{CV} \mathbf{V}\rho\, d\mathcal{V}\right) + \sum (\dot{m}_i \mathbf{V}_i)_{out} - \sum (\dot{m}_i \mathbf{V}_i)_{in}} \tag{3.40}$$

This is a commonly used approximation in engineering analyses. It is crucial to realize that we are dealing with vector sums. Equation (3.40) states that the net vector force on a fixed control volume equals the rate of change of vector momentum within the control volume plus the vector sum of outlet momentum fluxes minus the vector sum of inlet fluxes.

Net Pressure Force on a Closed Control Surface:

The external pressure force on a surface is normal to the surface & inward. Therefore,

$$\vec{F}_{pres} = \int_{CS} p(-\vec{n})\,dA$$

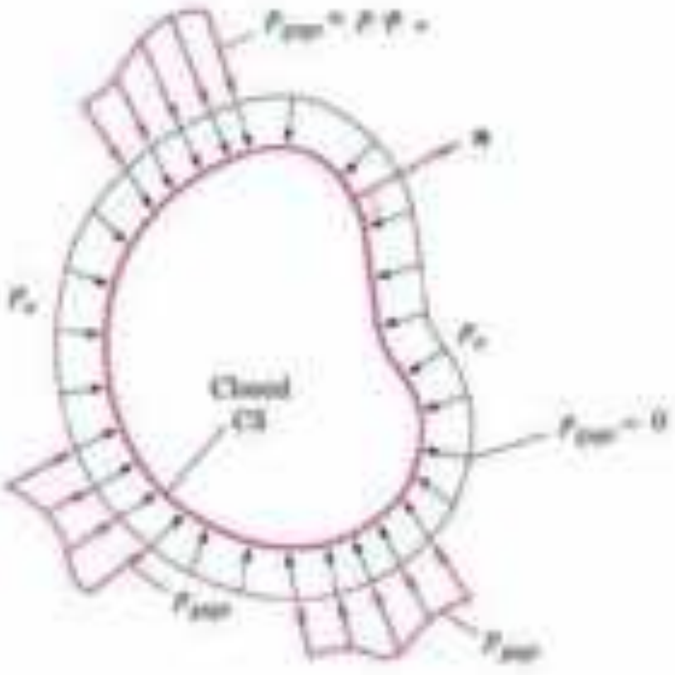


When the pressure has a uniform value over the surface:

$$\vec{F}_{up} = \int_{CS} p_a(-\vec{n})\,dA = -p_a\int \vec{n}\,dA$$

$$=0$$

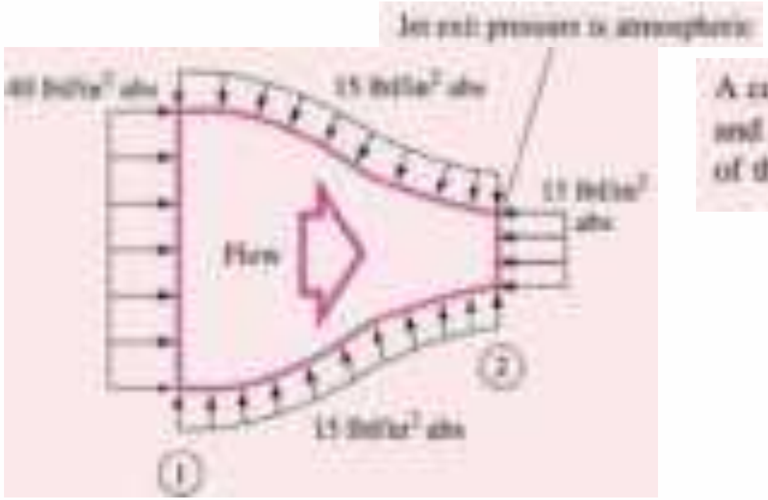
This result is independent of the shape of the surface as long as the surface is closed.



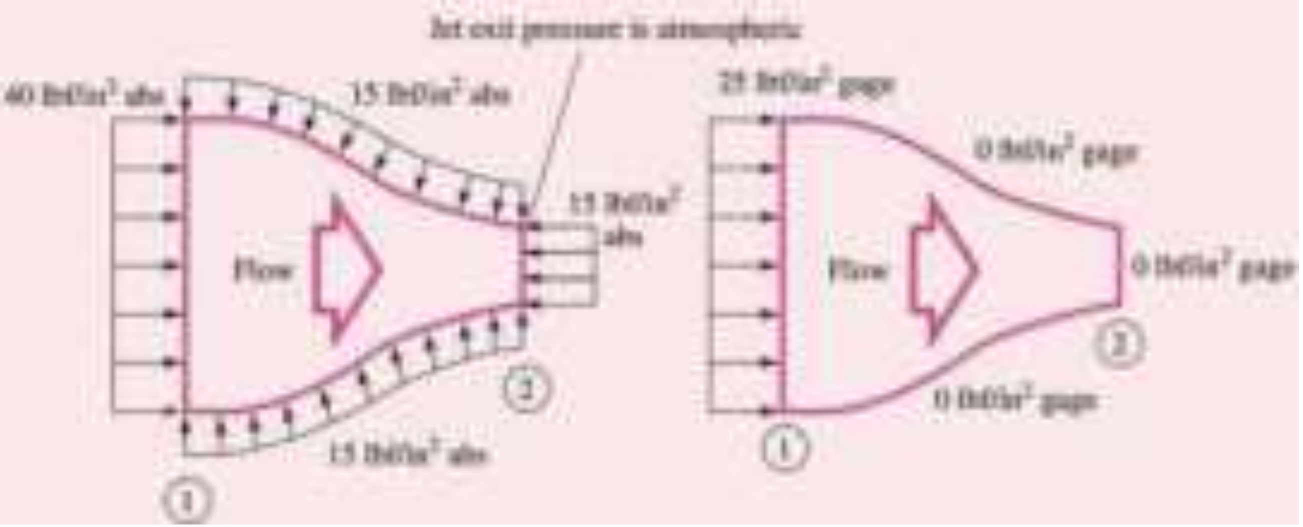
A seemingly complicated pressure force problem can be simplified by substracting any convenient uniform pressure pa & working only with the remaining gauge pressures.

$$\vec{F}_{press} = \int_{CS} (p - p_a)(-\vec{n})\,dA$$

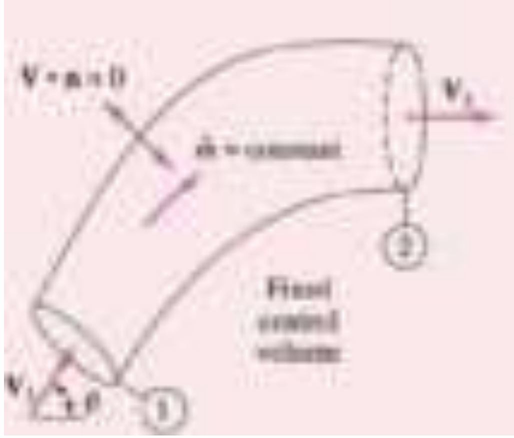
$$= \int_{CS} p_{gage}(-\vec{n})\,dA$$

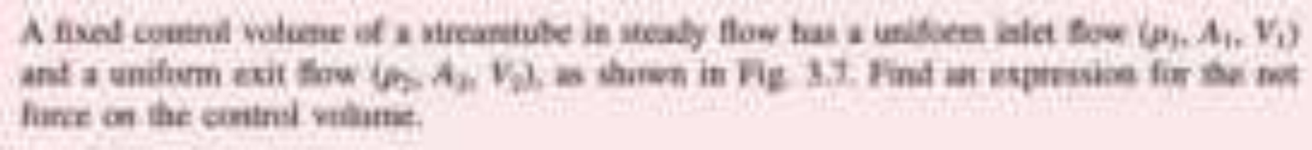


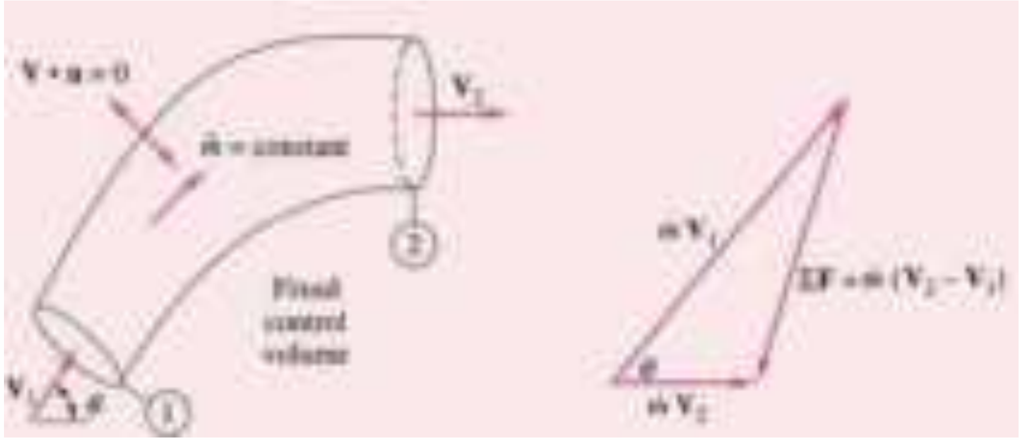
A control volume of a nozzle section has surface pressures of 40 lbf/in² absolute at section 1 and atmospheric pressure of 15 lbf/in² absolute at section 2 and on the external rounded part of the nozzle, as in Fig. E3.6a. Compute the net pressure force if $D_1 = 3$ in and $D_2 = 1$ in.



$$\mathbf{F}_{\text{press}} = P_{\text{gage},1}(-\mathbf{n})_1 A_1 = \left(25\,\frac{\text{lbf}}{\text{in}^2}\right)\left[-(-\mathbf{i})\right]\left[\frac{\pi}{4}(3\text{ in})^2\right] = 177\mathbf{i}\text{ lbf}$$




A fixed control volume of a streamtube in steady flow has a uniform inlet flow (ρ_1, A_1, V_1) and a uniform exit flow (ρ_2, A_2, V_2), as shown in Fig. 3.7. Find an expression for the net force on the control volume.

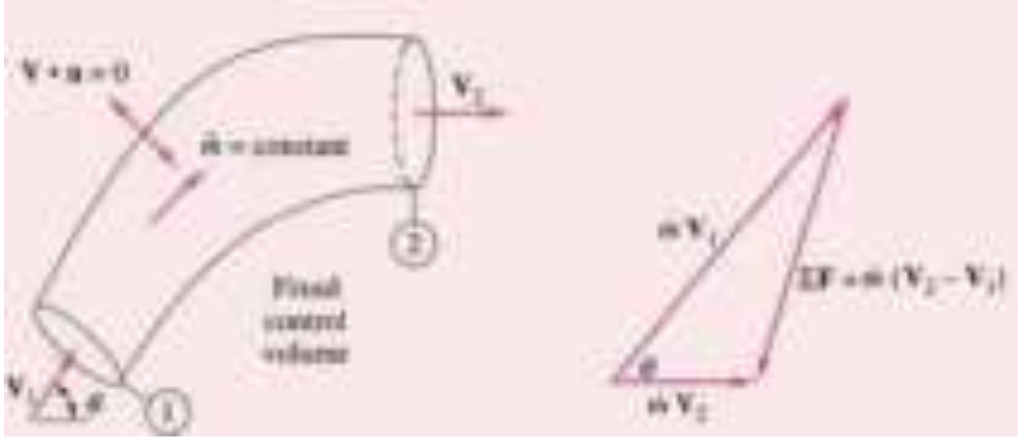


$$\sum \vec{F} = \frac{d}{dt}\int_{cv} \vec{v}\rho\, d\vartheta + \sum(m_i\vec{v}_i)_{out} - \sum(m_i\vec{v}_i)_{in}$$

$$\frac{d}{dt}\int_{cv} \vec{v}\rho\, d\vartheta = \int_{cv} \frac{\partial}{\partial t}(\vec{v}\rho)\, d\vartheta = 0$$

↙ constant volume

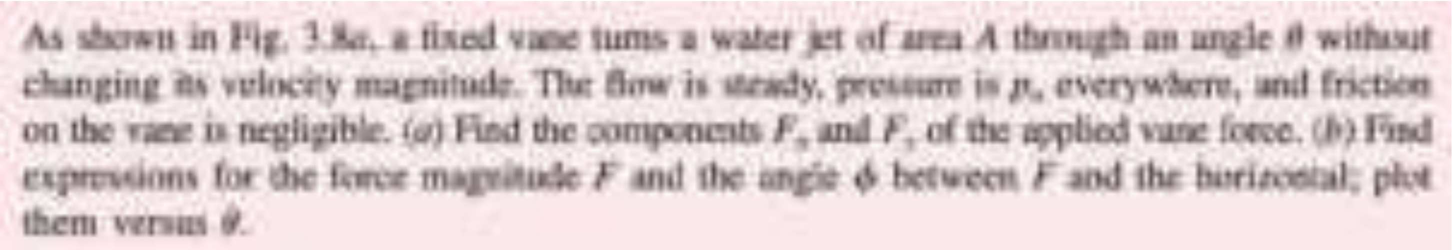
↙ steady flow : $\partial/\partial t \to 0$



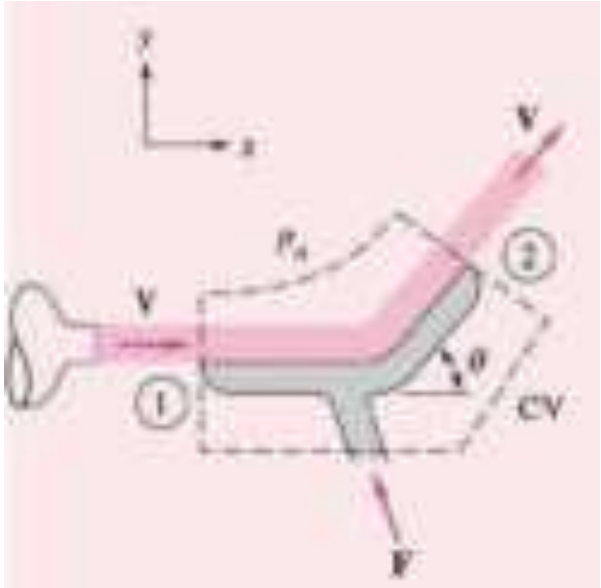
$$\sum \vec{F} = \dot{m}_2 \vec{V}_2 - \dot{m}_1 \vec{V}_1 = (\rho_2 A_2 V_2)\vec{V}_2 - (\rho_1 A_1 V_1)\vec{V}_1$$

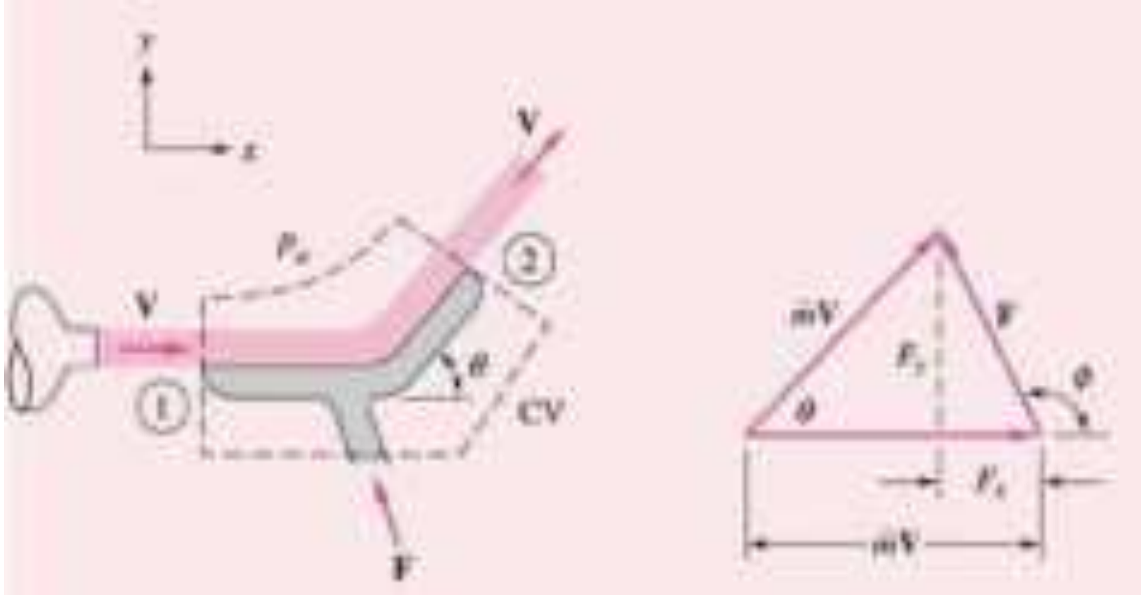
Cons. of. mass : $\dot{m}_1 = \dot{m}_2 = \dot{m} = \text{const}$

Therefore, $\sum \vec{F} = \dot{m}(\vec{V}_2 - \vec{V}_1)$



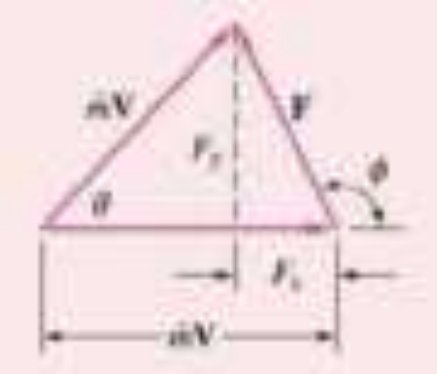
As shown in Fig. 3.8*a*, a fixed vane turns a water jet of area A through an angle θ without changing its velocity magnitude. The flow is steady, pressure is p_a everywhere, and friction on the vane is negligible. (*a*) Find the components F_x and F_y of the applied vane force. (*b*) Find expressions for the force magnitude F and the angle ϕ between F and the horizontal; plot them versus θ.





$\vec{F}_{vane} = \dot{m}_2 \vec{V}_2 - \dot{m}_1 \vec{V}_1$ $\qquad V_1 = V_2 = V$

Cons. of mass: $\dot{m}_1 = \dot{m}_2 = \dot{m} = \rho A V$



$$F_x = \dot{m}V(\cos\theta - 1) \qquad F_y = \dot{m}V\sin\theta$$

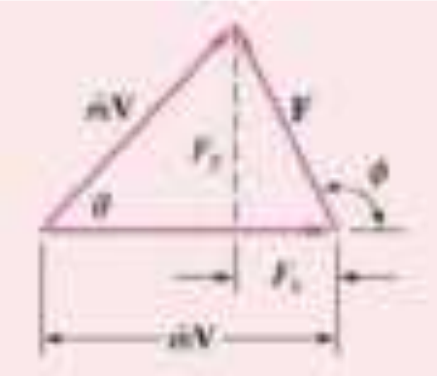
$$\dot{m}V = \rho A V^2$$

$$F = (F_x^2 + F_y^2)^{1/2} = \dot{m}V\left[\sin^2\theta + (\cos\theta - 1)^2\right]^{1/2}$$

$$= \dot{m}V\left[\sin^2\theta + \cos^2\theta + 1 - 2\cos\theta\right]^{1/2}$$

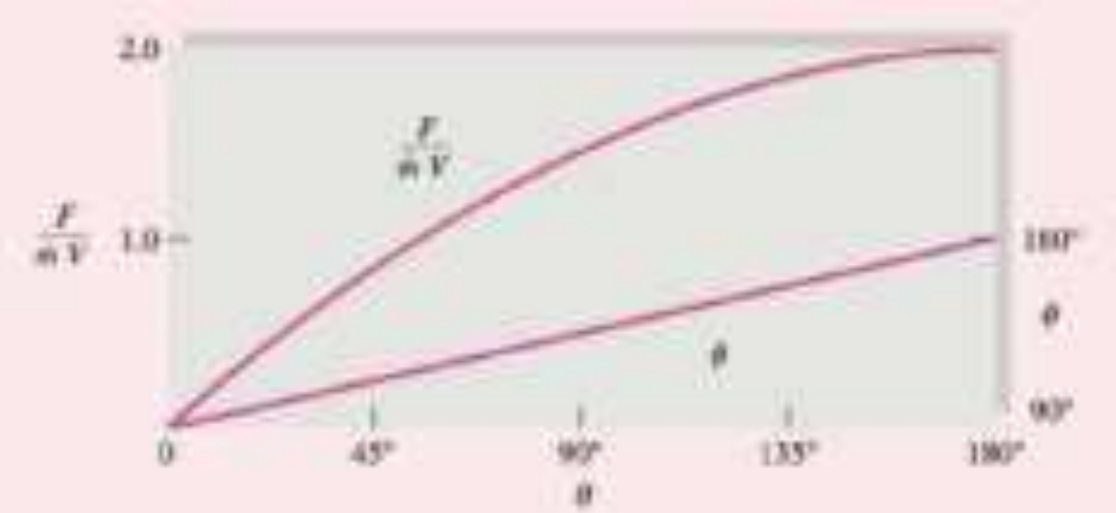
$$= \dot{m}V\left[2(1 - \cos\theta)\right]^{1/2}$$

$$= 2\dot{m}V\sin\frac{\theta}{2} \qquad \text{since } \sin^2\frac{\theta}{2} = \frac{1 - \cos\theta}{2}$$



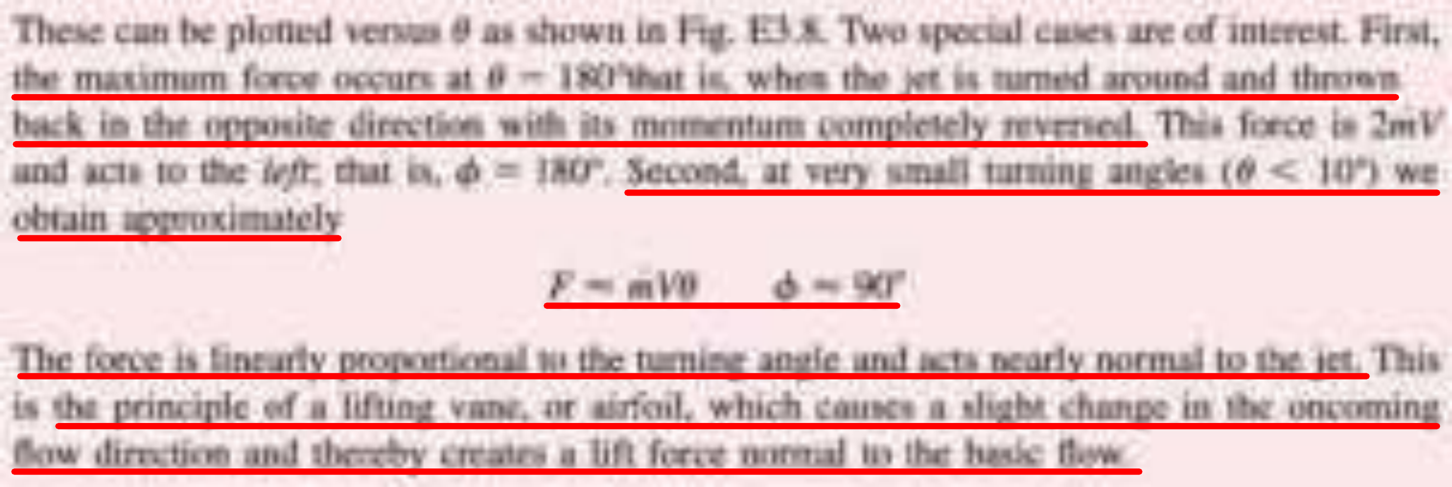
From geometry

$$\phi = 180° - \frac{180° - \theta}{2} = 90° + \frac{\theta}{2}$$



$$F = 2\dot{m}V\sin\frac{\theta}{2}$$

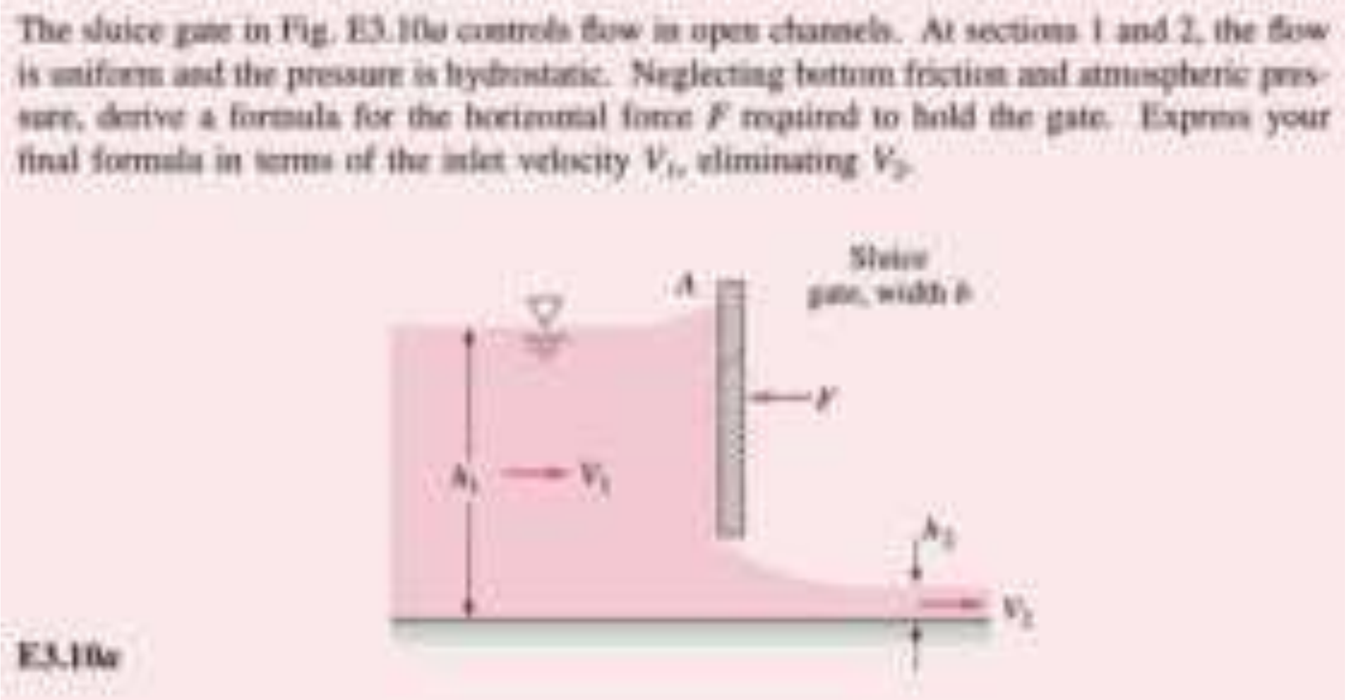
$$\phi = 90° + \frac{\theta}{2}$$



These can be plotted versus θ as shown in Fig. E3.8. Two special cases are of interest. First, the maximum force occurs at $\theta = 180°$that is, when the jet is turned around and thrown back in the opposite direction with its momentum completely reversed. This force is $2\dot{m}V$ and acts to the *left*; that is, $\phi = 180°$. Second, at very small turning angles ($\theta < 10°$) we obtain approximately

$$F \approx \dot{m}V\theta \qquad \phi \approx 90°$$

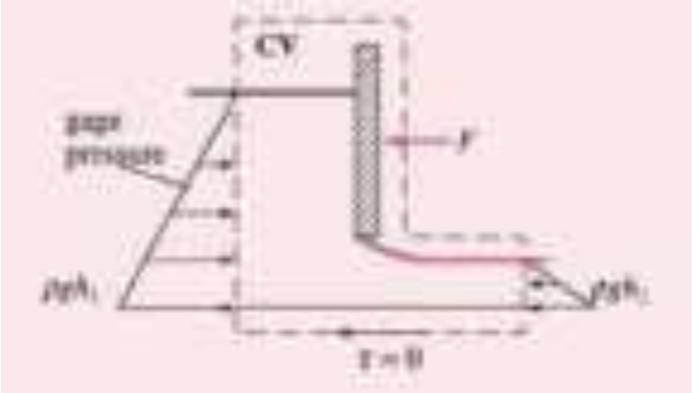
The force is linearly proportional to the turning angle and acts nearly normal to the jet. This is the principle of a lifting vane, or airfoil, which causes a slight change in the oncoming flow direction and thereby creates a lift force normal to the basic flow.



The sluice gate in Fig. E3.10a controls flow in open channels. At sections 1 and 2, the flow is uniform and the pressure is hydrostatic. Neglecting bottom friction and atmospheric pressure, derive a formula for the horizontal force F required to hold the gate. Express your final formula in terms of the inlet velocity V_1, eliminating V_2.

A
Sluice gate, width b
F
h1
V1
h2
V2

E3.10a



Assume steady incompressible flow with no variation across the width b.

$$\dot{m} = \rho_1 V_1 h_1 b = \rho_2 V_2 h_2 b \ , \ \rho_1 = \rho_2 = \rho$$

$$\Rightarrow \quad V_2 = \frac{h_1}{h_2} V_1$$

we can use gauge pressures since a uniform atmospheric pressure causes no net force.

$$\Sigma F_x = -F_{gate} + \frac{1}{2} \rho g h_1 (h_1 b) - \frac{1}{2} \rho g h_2 (h_2 b) = \dot{m} (V_2 - V_1)$$

$$\dot{m} = \rho h_1 b V_1$$

Hw:

$$F_{gate} = \frac{\rho}{2} g b h_1^2 \left[1 - \left(\frac{h_2}{h_1} \right)^2 \right] - \rho h_1 b V_1^2 \left(\frac{h_1}{h_2} - 1 \right)$$

Momentum Flux Correction Factor:

For a flow in a duct, the axial velocity is usually non-uniform.

When uniform: $\int u \rho (\vec{V} \cdot \vec{n}) dA = \dot{m} V = \rho A V^2$

For a non-uniform flow, this should be corrected to

$$\int u \rho (\vec{V} \cdot \vec{n}) dA = \beta \rho A V^2$$ where

$\beta \geqslant 1$: dimensionless momentum flux correction factor

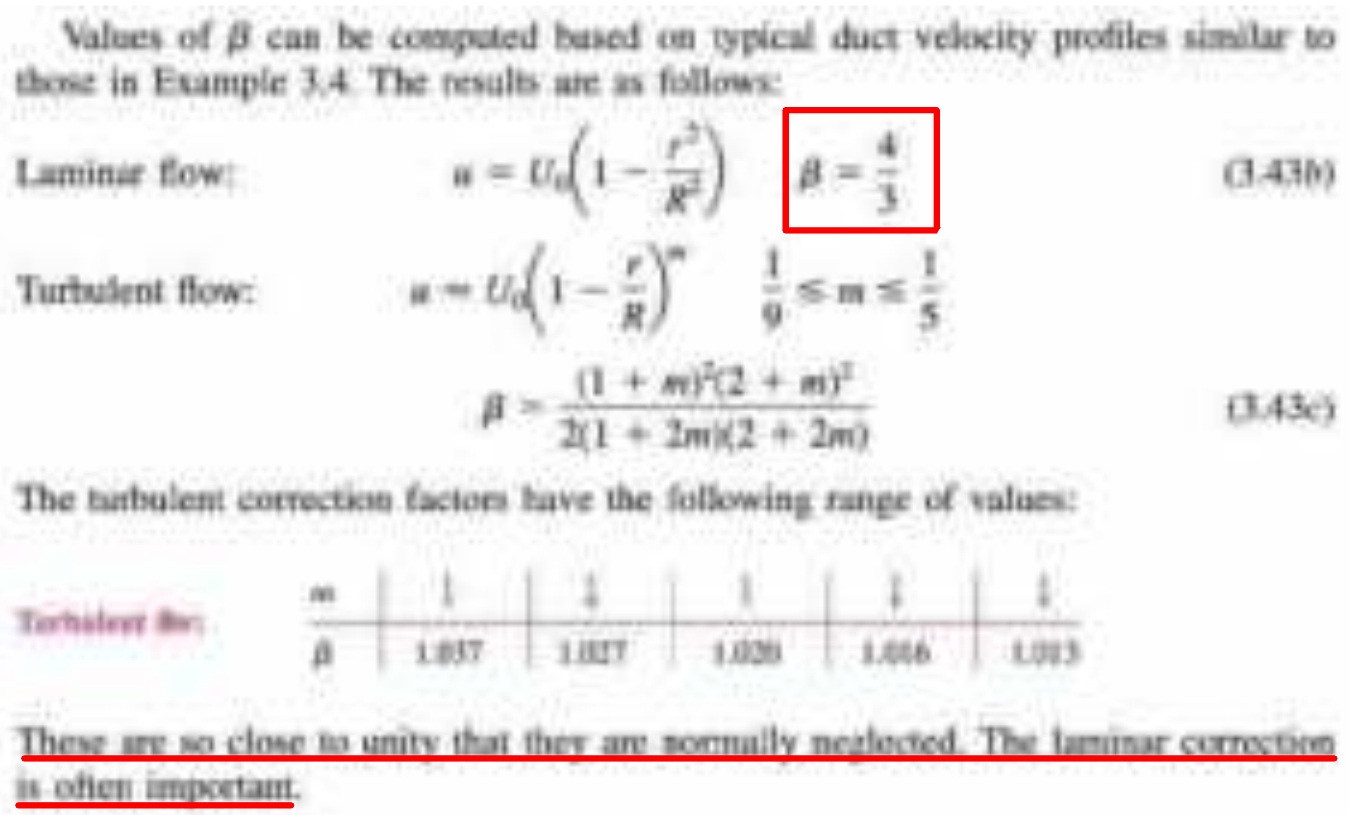
β: accounts for the variation of the velocity across the duct section.

To find it, we compute the exact flux & set it equal to a flux based on average velocity in the duct:

$$\rho \int u^2 dA = \beta \dot{m} V_{av} = \beta \rho A V_{av}^2 \; ; \; V_{av} = \frac{1}{A}\int u \, dA$$

Therefore,

$$\beta = \frac{1}{A}\int \left(\frac{u}{V_{av}}\right)^2 dA$$



Values of β can be computed based on typical duct velocity profiles similar to those in Example 3.4. The results are as follows:

Laminar flow: $$u = U_0\left(1 - \frac{r^2}{R^2}\right) \qquad \boxed{\beta = \frac{4}{3}} \tag{3.43b}$$

Turbulent flow: $$u \approx U_0\left(1 - \frac{r}{R}\right)^m \qquad \frac{1}{9} \le m \le \frac{1}{5}$$

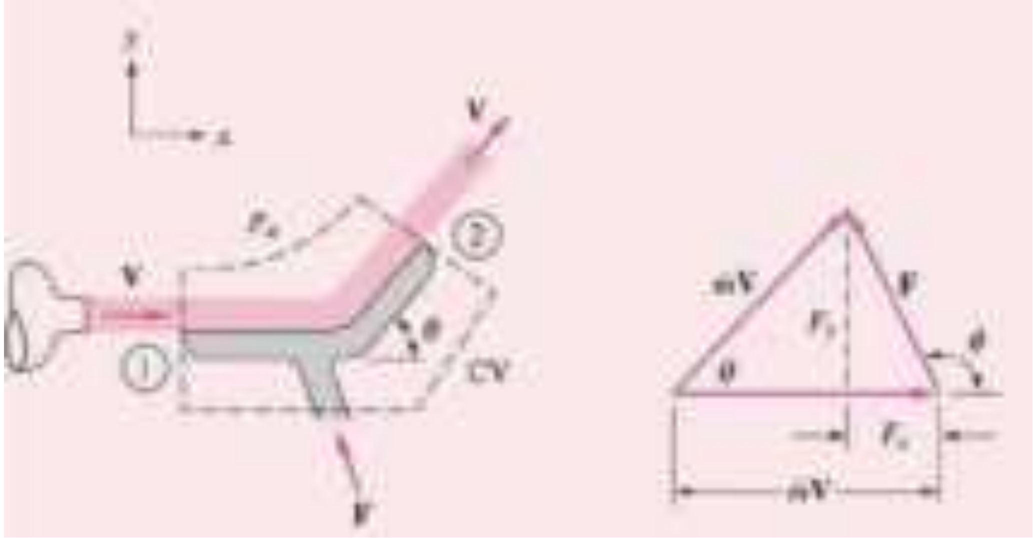
$$\beta = \frac{(1+m)^2(2+m)^2}{2(1+2m)(2+2m)} \tag{3.43c}$$

The turbulent correction factors have the following range of values:

Turbulent flow:

m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
β	1.037	1.027	1.020	1.016	1.013

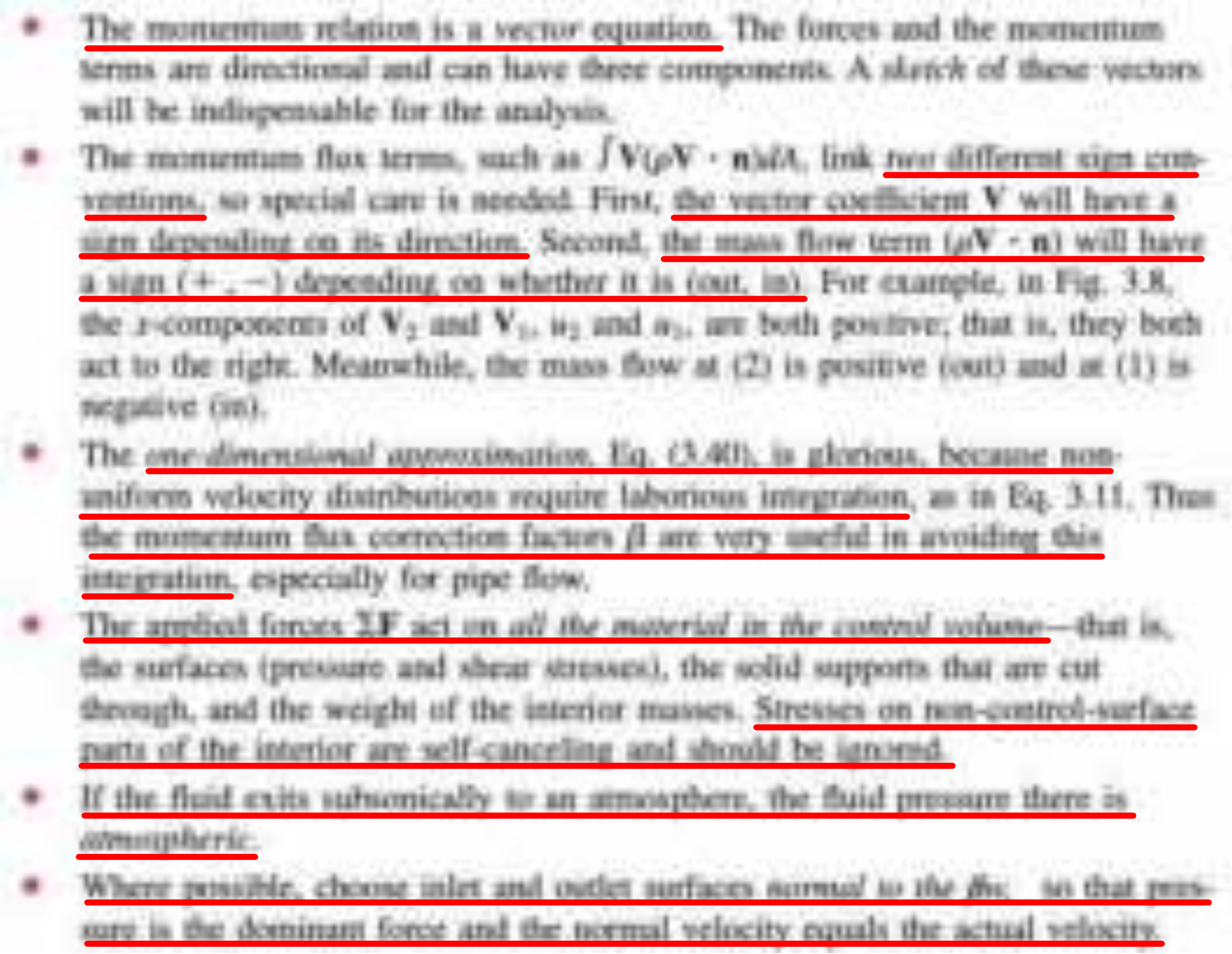
These are so close to unity that they are normally neglected. The laminar correction is often important.



For example, when we have nonuniform velocities at sections 1 & 2, we should write

$$\sum \vec{F} = \dot{m}\left(\beta_2 \vec{V}_2 - \beta_1 \vec{V}_1\right)$$

Linear momentum Tips



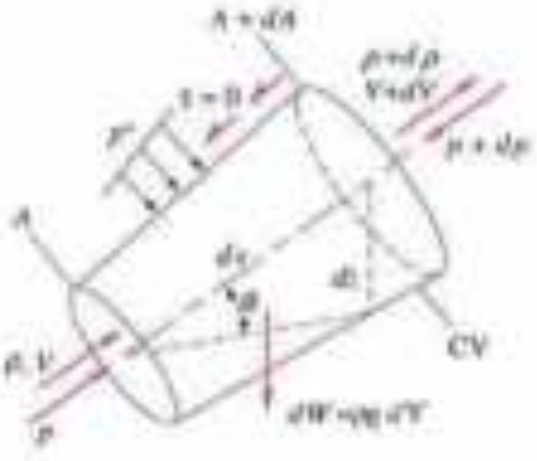
- The momentum relation is a vector equation. The forces and the momentum terms are directional and can have three components. A *sketch* of these vectors will be indispensable for the analysis.
- The momentum flux terms, such as $\int \mathbf{V}(\rho\mathbf{V}\cdot\mathbf{n})dA$, link *two different sign conventions*, so special care is needed. First, the vector coefficient **V** will have a sign depending on its direction. Second, the mass flow term $(\rho\mathbf{V}\cdot\mathbf{n})$ will have a sign (+, −) depending on whether it is (out, in). For example, in Fig. 3.8, the x-components of $\mathbf{V}_2$ and $\mathbf{V}_1$, u_2 and u_1, are both positive; that is, they both act to the right. Meanwhile, the mass flow at (2) is positive (out) and at (1) is negative (in).
- The *one-dimensional approximation*, Eq. (3.40), is glorious, because nonuniform velocity distributions require laborious integration, as in Eq. 3.11. Thus the momentum flux correction factors β are very useful in avoiding this integration, especially for pipe flow.
- The applied forces $\Sigma\mathbf{F}$ act on *all the material in the control volume*—that is, the surfaces (pressure and shear stresses), the solid supports that are cut through, and the weight of the interior masses. Stresses on non-control-surface parts of the interior are self-canceling and should be ignored.
- If the fluid exits subsonically to an atmosphere, the fluid pressure there is *atmospheric*.
- Where possible, choose inlet and outlet surfaces *normal to the flow*, so that pressure is the dominant force and the normal velocity equals the actual velocity.

3.5) The Frictionless Flow: The Bernoulli Eqn.

A classic linear momentum relation is a relation between pressure, velocity & elevation in a frictionless flow, now called the Bernoulli eqn.

It is very famous & widely used but one should be wary of its restrictions: All fluids are viscous.

We can use it for regions of the flow that are nearly frictionless.



Consider an elemental fixed streamtube control volume of variable area A(s) and length ds

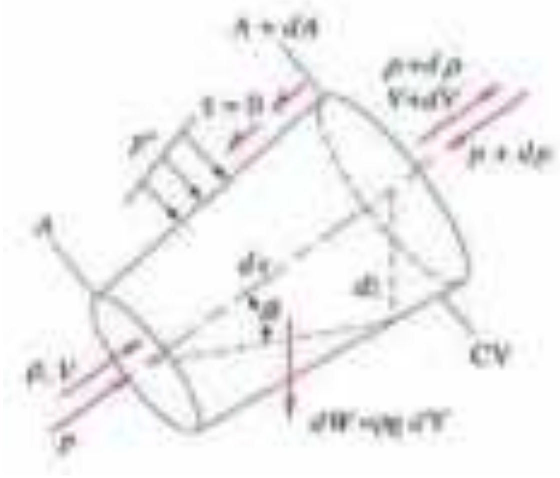
S: streamline direction

Properties (ρ, A, V) may vary with s but assumed to be uniform over A.

The streamtube orientation θ is arbitrary.

Elevation change: $dz = ds \sin\theta$

Friction on the streamtube walls are neglected.
⇒ a very restrictive assumption.



The limit of a vanishingly small area means that the streamtube is equivalent to a streamline of the flow.

Bernoulli eqn. is valid for both & is usually stated as holding "along a streamline" in frictionless flow.

Cons. of mass: $\frac{d}{dt}\int \rho \, d\mathcal{V} + \dot{m}_{out} - \dot{m}_{in} = 0 \approx \frac{\partial \rho}{\partial t} d\mathcal{V} + d\dot{m}$

where $\dot{m} = \rho A V$ and $d\mathcal{V} \approx A\,ds$. Therefore,

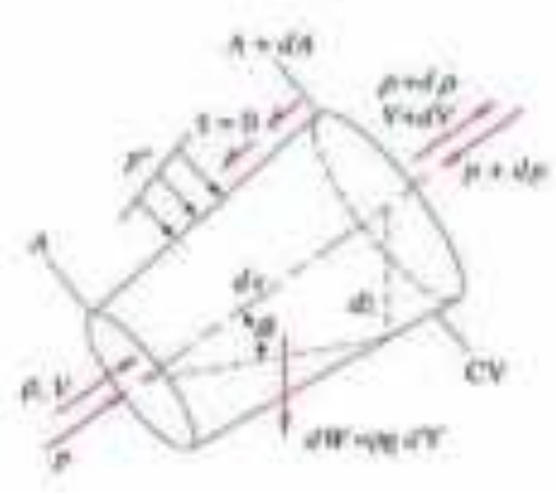
$$d\dot{m} = d(\rho A V) = -\frac{\partial \rho}{\partial t} A\,ds$$

Linear momentum relation in the streamwise direction:

$$\sum F_s = \frac{d}{dt}\int_{CV} V \rho \, d\vartheta + (\dot{m}V)_{out} - (\dot{m}V)_{in}$$

$$\approx \frac{\partial}{\partial t}(\rho V) A \, ds + d(\dot{m}V)$$

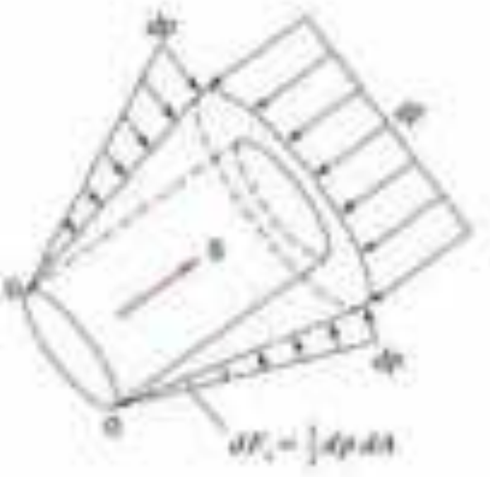
$V_s = V$ since s is the streamline direction



$$dF_{s,grav} = -dW \sin\theta$$

$$= -\gamma A \, ds \sin\theta$$

$$= -\gamma A \, dz$$



For the pressure force, we first substract a uniform ralue p from all surfaces (the net force is not changed.)

The pressure along the slanted side of the streamtube has a streamwise component that acts on the outer ring of area increase dA: $dF_s \approx \frac{1}{2} dp\, dA$

The net pressure force:

$$dF_{s,press} = \frac{1}{2} dp\, dA - dp\,(A + dA) \approx -A\,dp$$

to the first order in infisimal quantities.

$$\Sigma dF_s = -\gamma A dz - A dp = \frac{\partial}{\partial t}(\rho V) A ds + d(\dot{m} V)$$

$$= \frac{\partial \rho}{\partial t} \rho V A ds + \frac{\partial V}{\partial t} \rho A ds + \dot{m} dV + V d\dot{m}$$

Remember the continuity eqn: $d\dot{m} = -\frac{\partial \rho}{\partial t} A ds$

⇒ 1st & the last term cancel each other.

$$-\gamma A dz - A dp = \frac{\partial V}{\partial t} \rho A ds + \dot{m} d\bar{V}$$

Divide by ρA & rearrange:

$$\boxed{\frac{\partial V}{\partial t}ds + \frac{dp}{\rho} + VdV + g\,dz = 0}$$

Bernoulli eqn. for unsteady frictionless flow along a streamline.

We can integrate it between any two points 1 & 2 on the streamline:

$$\int_1^2 \frac{\partial V}{\partial t}ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0$$

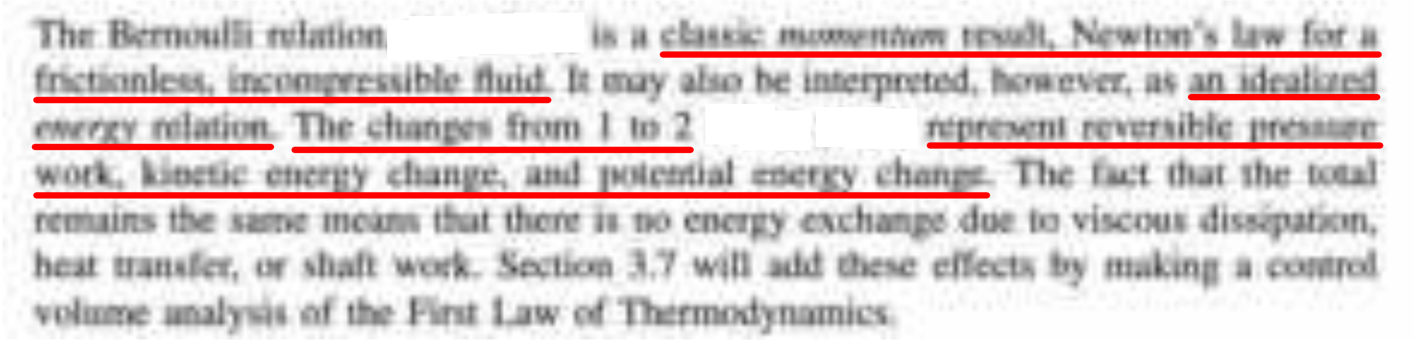
Steady Incompressible Flow: $\frac{\partial V}{\partial t} = 0 \qquad \rho = \text{const.}$

$$\int_1^2 \cancel{\frac{\partial V}{\partial t}} ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0$$

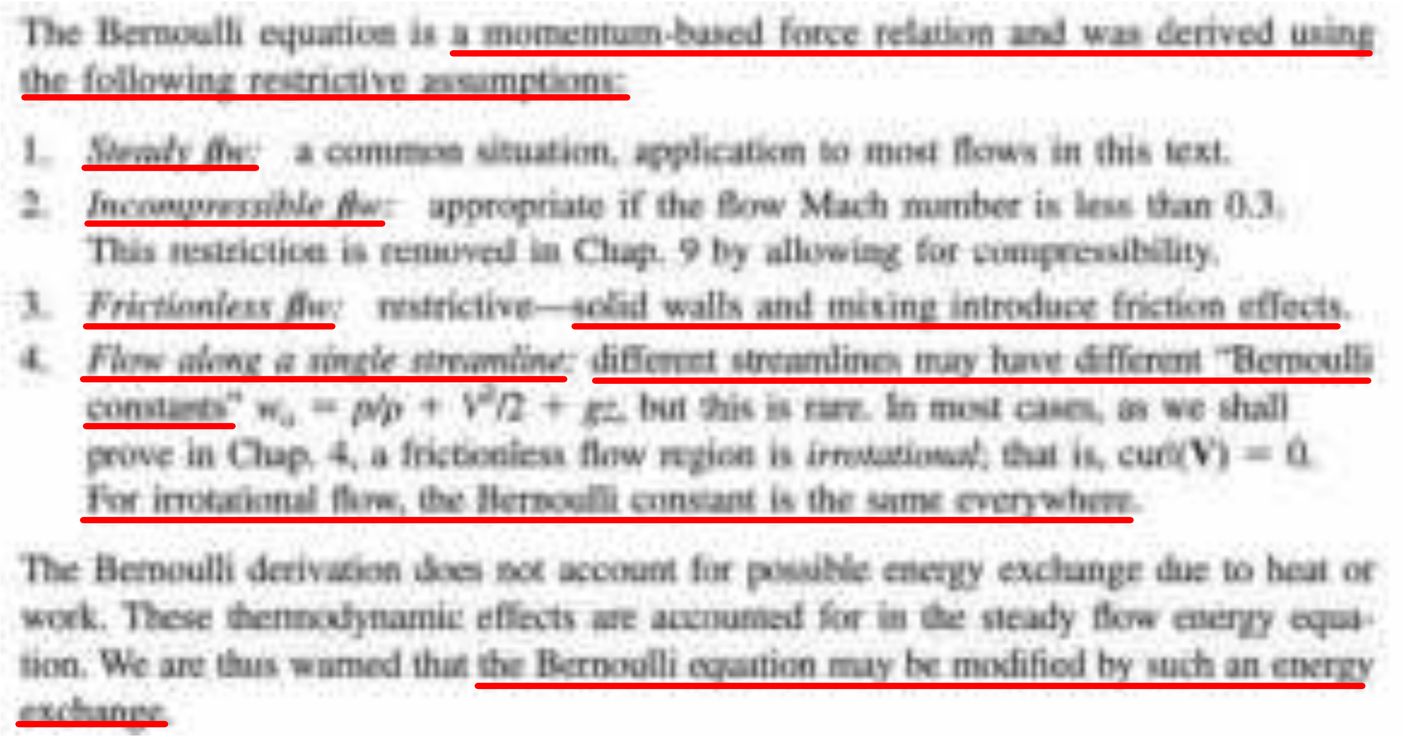
$$\frac{p_2 - p_1}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g\left(z_2 - z_1\right) = 0 \qquad \text{or}$$

$$\boxed{\frac{p_1}{\rho} + \frac{1}{2}V_1^2 + g z_1 = \frac{p_2}{\rho} + \frac{1}{2}V_2^2 + g z_2 = \text{const}}$$

Bernoulli Interpreted as an Energy Relation:



The Bernoulli relation is a classic *momentum* result, Newton's law for a frictionless, incompressible fluid. It may also be interpreted, however, as *an idealized energy relation*. The changes from 1 to 2 represent reversible pressure work, kinetic energy change, and potential energy change. The fact that the total remains the same means that there is no energy exchange due to viscous dissipation, heat transfer, or shaft work. Section 3.7 will add these effects by making a control volume analysis of the First Law of Thermodynamics.



The Bernoulli equation is a momentum-based force relation and was derived using the following restrictive assumptions:

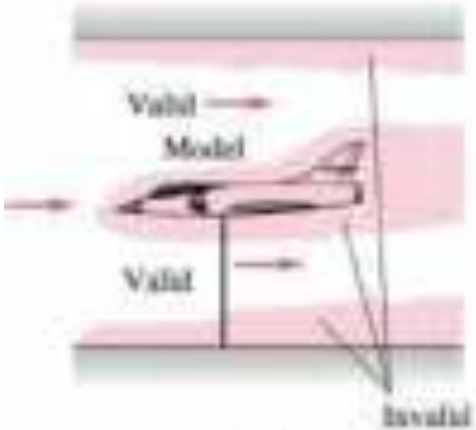
1. *Steady flow:* a common situation, application to most flows in this text.
2. *Incompressible flow:* appropriate if the flow Mach number is less than 0.3. This restriction is removed in Chap. 9 by allowing for compressibility.
3. *Frictionless flow:* restrictive—solid walls and mixing introduce friction effects.
4. *Flow along a single streamline:* different streamlines may have different "Bernoulli constants" $w_0 = p/\rho + V^2/2 + gz$, but this is rare. In most cases, as we shall prove in Chap. 4, a frictionless flow region is *irrotational;* that is, curl(**V**) = 0. For irrotational flow, the Bernoulli constant is the same everywhere.

The Bernoulli derivation does not account for possible energy exchange due to heat or work. These thermodynamic effects are accounted for in the steady flow energy equation. We are thus warned that the Bernoulli equation may be modified by such an energy exchange.

$M = \frac{u}{c}$

u: flow velocity

c: speed of sound in the medium



(a)

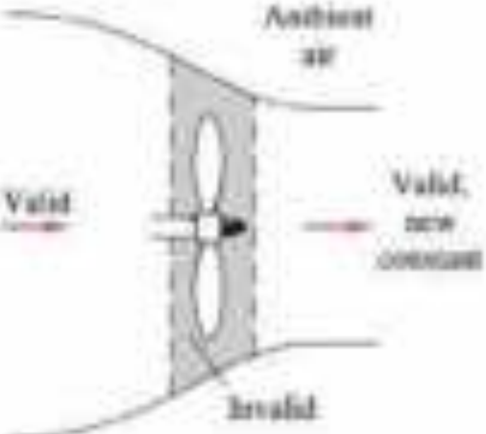
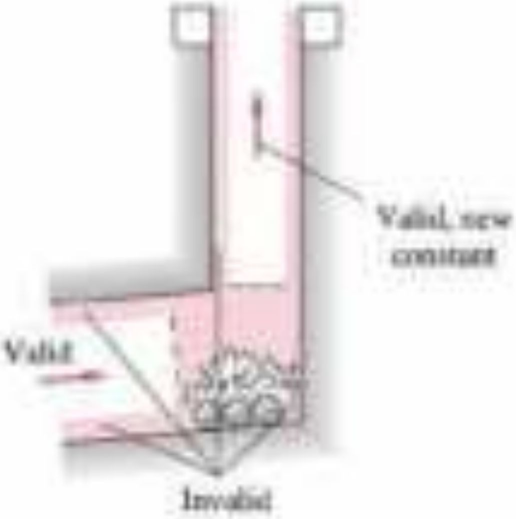


Figure 3.13 illustrates some practical limitations on the use of Bernoulli's equation (3.54). For the wind tunnel model test of Fig. 3.13*a*, the Bernoulli equation is valid in the core flow of the tunnel but not in the tunnel wall boundary layers, the model surface boundary layers, or the wake of the model, all of which are regions with high friction.

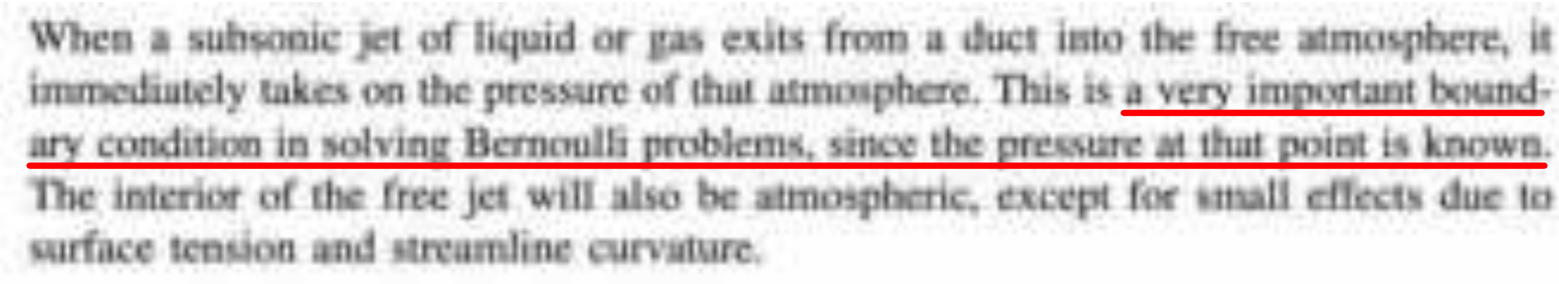
In the propeller flow of Fig. 3.13*b*, Bernoulli's equation is valid both upstream and downstream, but with a different constant $w_0 = p/\rho + V^2/2 + gz$, caused by the work addition of the propeller. The Bernoulli relation (3.54) is not valid near the propeller blades or in the helical vortices (not shown, see Fig. 1.14) shed downstream of the blade edges. Also, the Bernoulli constants are higher in the flowing "slipstream" than in the ambient atmosphere because of the slipstream kinetic energy.

For the chimney flow of Fig. 3.13*c*, Eq. (3.54) is valid before and after the fire, but with a change in Bernoulli constant that is caused by heat addition. The Bernoulli equation is not valid within the fire itself or in the chimney wall boundary layers.



(c)

Jet Exit Pressure Equals Atmospheric Pressure:



When a subsonic jet of liquid or gas exits from a duct into the free atmosphere, it immediately takes on the pressure of that atmosphere. This is a very important boundary condition in solving Bernoulli problems, since the pressure at that point is known. The interior of the free jet will also be atmospheric, except for small effects due to surface tension and streamline curvature.

Stagnation, Static & Dynamic Pressures:

In many incompressible-flow Bernoulli analyses, elevation changes are negligible. In such a case, we write

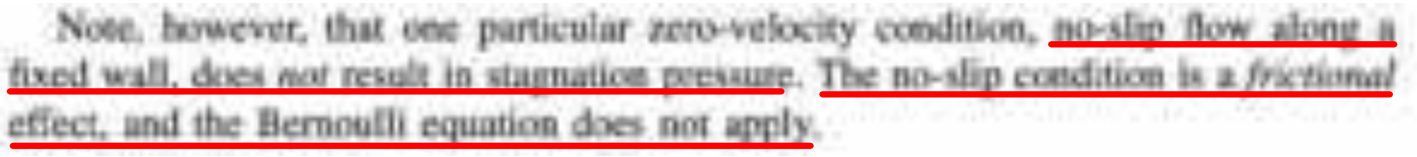
$$P_1 + \frac{1}{2} \rho V_1^2 = P_2 + \frac{1}{2} \rho V_2^2 = P_0 = \text{const.}$$

P_0: pressure at any point in the frictionless flow where the velocity is zero. It is called the stagnation pressure.

$$P_1 + \frac{1}{2}\rho V_1^2 = P_2 + \frac{1}{2}\rho V_2^2 = P_0$$

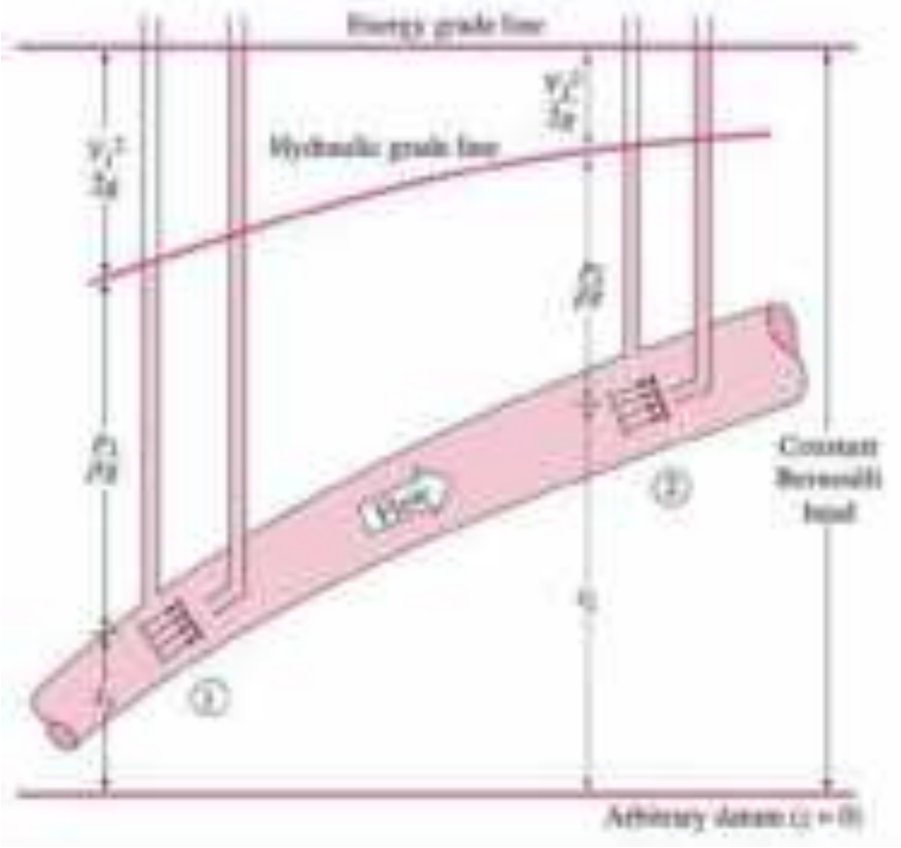
P_1, P_2: static pressures

$\frac{1}{2}\rho V^2$: dynamic pressure



Note, however, that one particular zero-velocity condition, no-slip flow along a fixed wall, does *not* result in stagnation pressure. The no-slip condition is a *frictional* effect, and the Bernoulli equation does not apply.

Hydraulic & Energy Grade Lines:

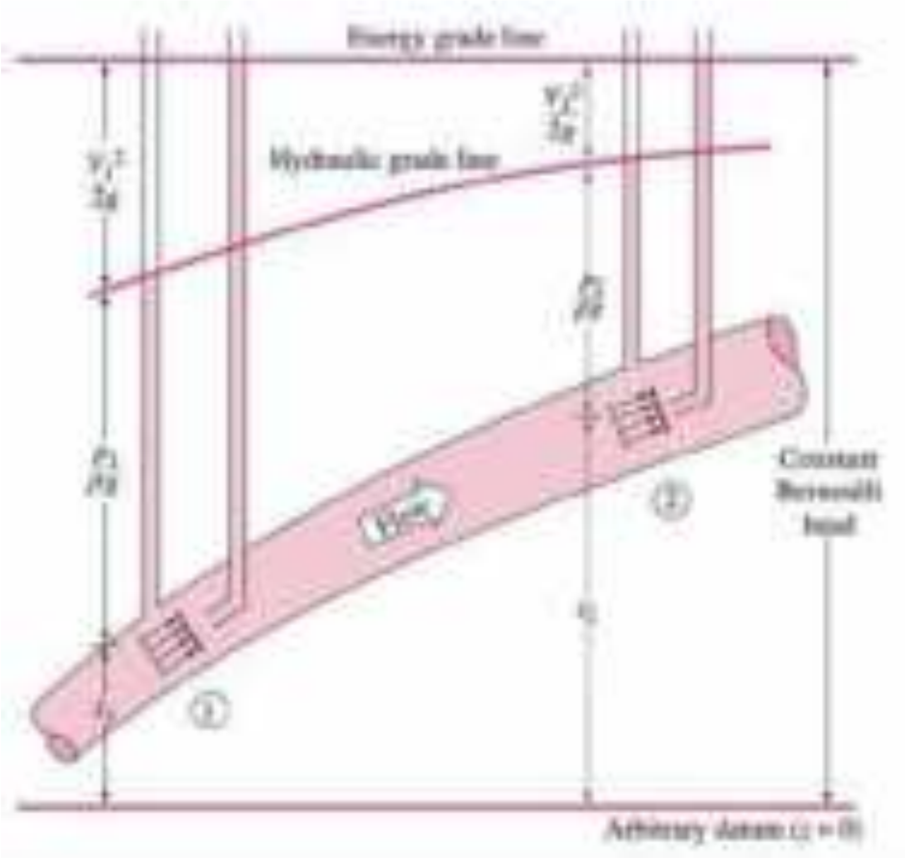


Energy Grade Line (EGL): the height of the total Bernoulli Constant

$$\text{EGL}: h_0 = z + \frac{P}{\gamma} + \frac{V^2}{2g}$$

Frictionless flow: No work or heat transfer ⇒ EGL has a constant height

$\gamma = \rho g$: spesific gravity

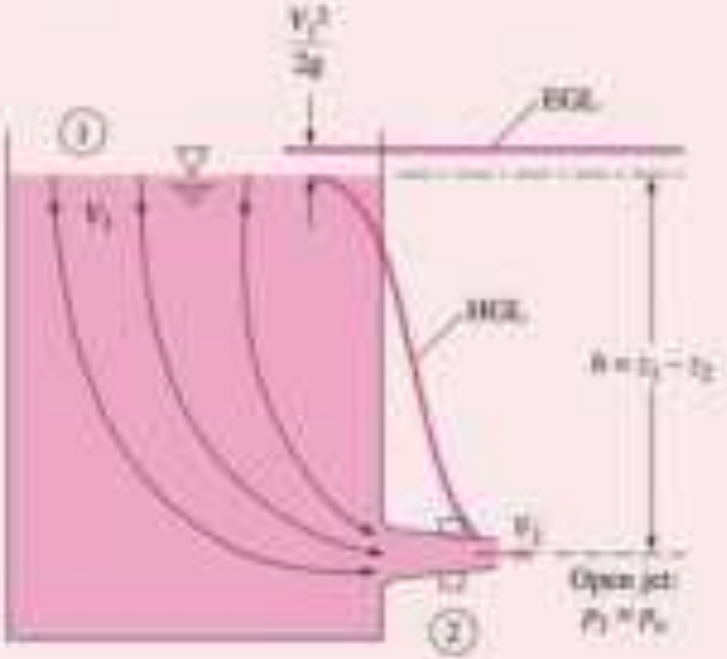


Hydraulic Grade Line (HGL): the height corresponding to elevation & pressure head

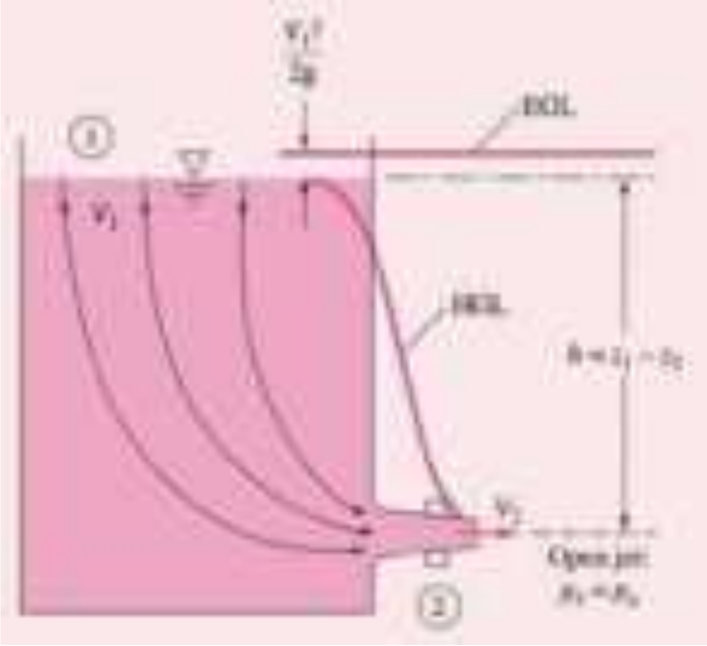
HGL: $z + \frac{P}{\gamma}$

$EGL - HGL = \frac{v^2}{2g}$: the velocity head

Find a relation between nozzle discharge velocity V_2 and tank free surface height h as in Fig. E3.13. Assume steady frictionless flow.



E3.13



1: upstream 2: downstream

A_1: tank cross section A_2: nozzle area

mass cons.: $A_1 V_1 = A_2 V_2$

Bernoulli eqn.: $\frac{p_1}{\rho} + \frac{V_1^2}{2g} + g z_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2g} + g z_2$

Both sections are subjected to the atmospheric pressure: $p_1 = p_2 = p_a$ ⇒ pressure terms cancel.

$$V_2^2 - V_1^2 = 2g(z_1 - z_2) = 2gh$$

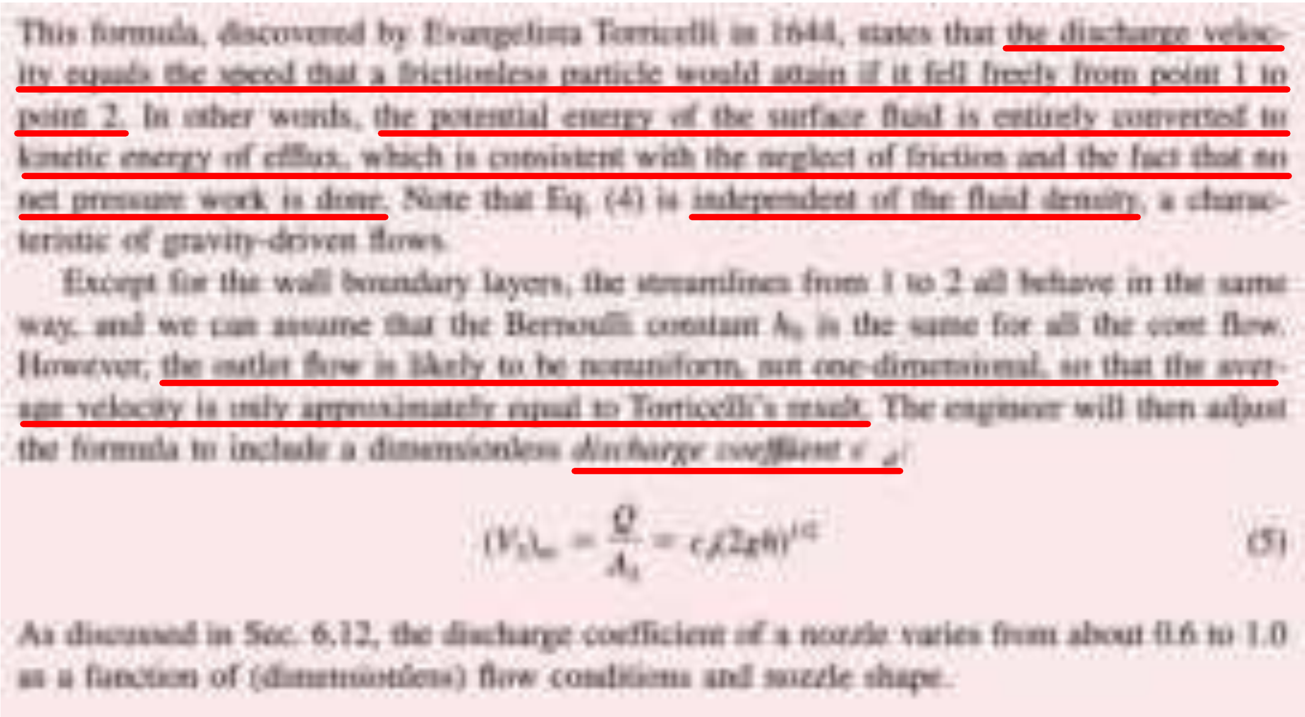
$$A_1 V_1 = A_2 V_2 \Rightarrow V_1 = \frac{A_2}{A_1} V_2$$

$$V_2^2 - V_1^2 = V_2^2\left[1 - \left(\frac{A_2}{A_1}\right)^2\right] = 2gh$$

$$V_2^2 = \frac{2gh}{1 - A_2^2/A_1^2}$$

Generally, A_2 is much smaller than A_1: $A_2/A_1 \approx 0$

Therefore, $V_2 \approx 2gh \Rightarrow$ an accurate approximation for the outlet velocity.



This formula, discovered by Evangelista Torricelli in 1644, states that the discharge velocity equals the speed that a frictionless particle would attain if it fell freely from point 1 to point 2. In other words, the potential energy of the surface fluid is entirely converted to kinetic energy of efflux, which is consistent with the neglect of friction and the fact that no net pressure work is done. Note that Eq. (4) is independent of the fluid density, a characteristic of gravity-driven flows.

Except for the wall boundary layers, the streamlines from 1 to 2 all behave in the same way, and we can assume that the Bernoulli constant h_0 is the same for all the core flow. However, the outlet flow is likely to be nonuniform, not one-dimensional, so that the average velocity is only approximately equal to Torricelli's result. The engineer will then adjust the formula to include a dimensionless *discharge coefficient* c_d:

$$(V_2)_{av} = \frac{Q}{A_2} = c_d(2gh)^{1/2} \tag{5}$$

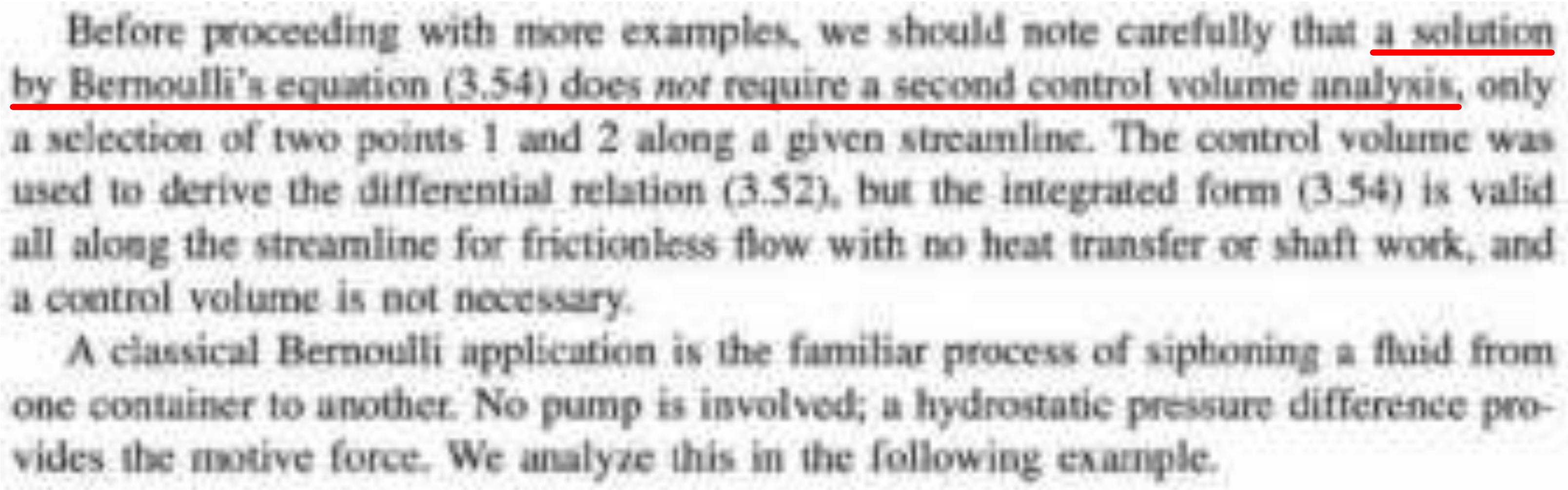
As discussed in Sec. 6.12, the discharge coefficient of a nozzle varies from about 0.6 to 1.0 as a function of (dimensionless) flow conditions and nozzle shape.

Surface Velocity Condition for a Large Tank:

Many problems involve liquid flow from a very large tank or reservoir. If the outflow is small compared to the volume of the tank, the surface of the tank hardly moves!

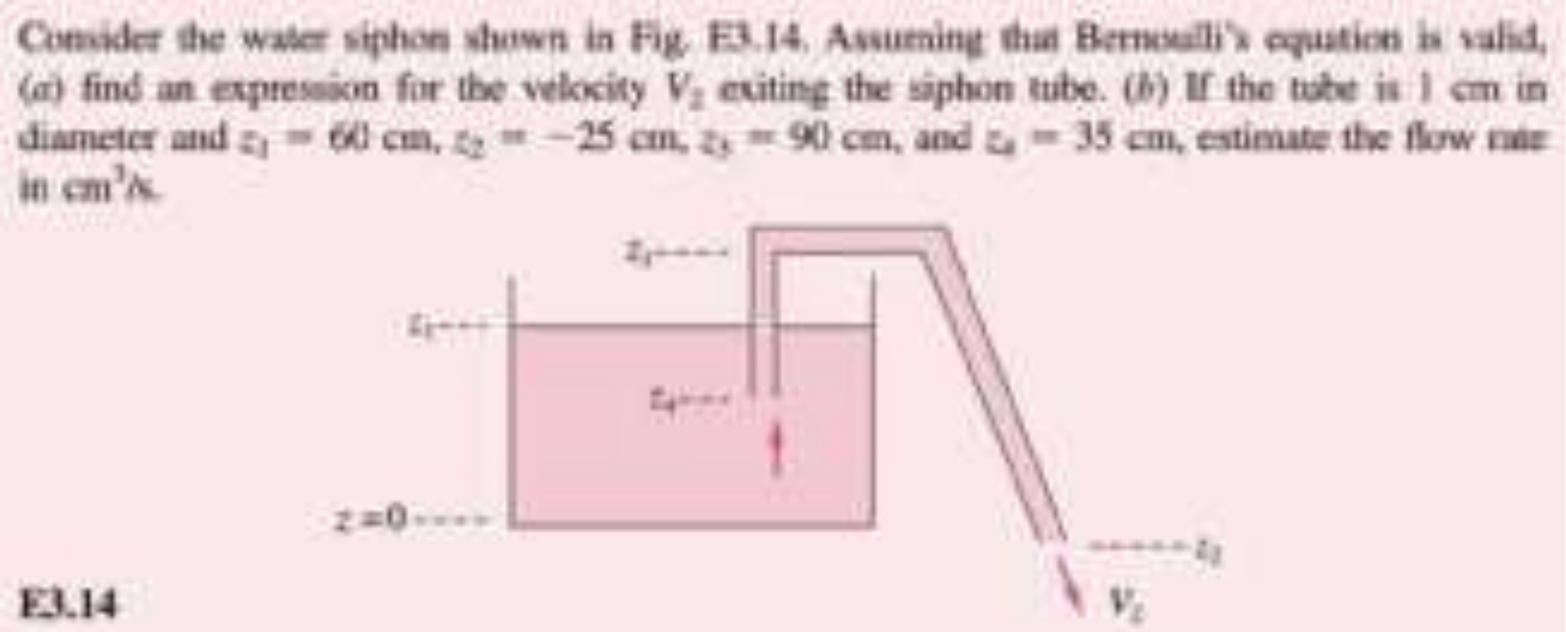
These problems are analyzed assuming zero velocity at the tank surface.

The pressure at the top surface at the top of the tank or reservoir is assumed to be atmospheric.



Before proceeding with more examples, we should note carefully that a solution by Bernoulli's equation (3.54) does *not* require a second control volume analysis, only a selection of two points 1 and 2 along a given streamline. The control volume was used to derive the differential relation (3.52), but the integrated form (3.54) is valid all along the streamline for frictionless flow with no heat transfer or shaft work, and a control volume is not necessary.

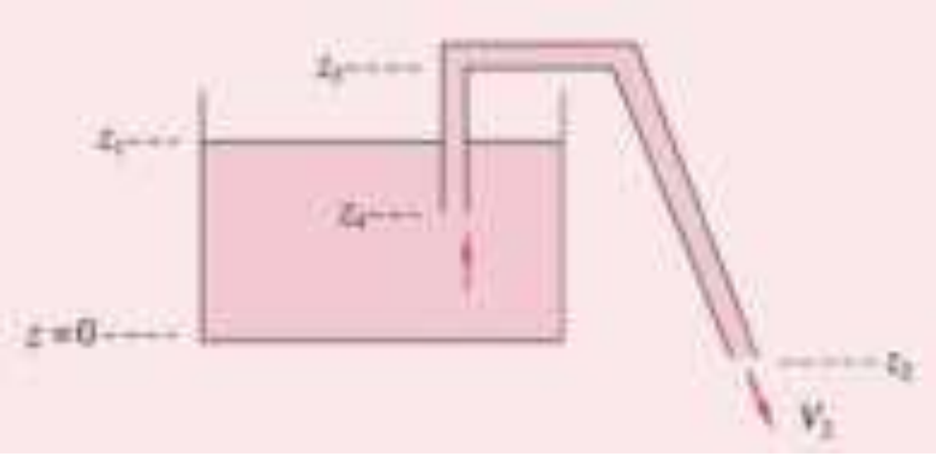
A classical Bernoulli application is the familiar process of siphoning a fluid from one container to another. No pump is involved; a hydrostatic pressure difference provides the motive force. We analyze this in the following example.



Consider the water siphon shown in Fig. E3.14. Assuming that Bernoulli's equation is valid, (*a*) find an expression for the velocity V_2 exiting the siphon tube. (*b*) If the tube is 1 cm in diameter and $z_1 = 60$ cm, $z_2 = -25$ cm, $z_3 = 90$ cm, and $z_4 = 35$ cm, estimate the flow rate in cm^3/s.

z_3
z_1
z_4
$z = 0$
z_2
V_2

E3.14



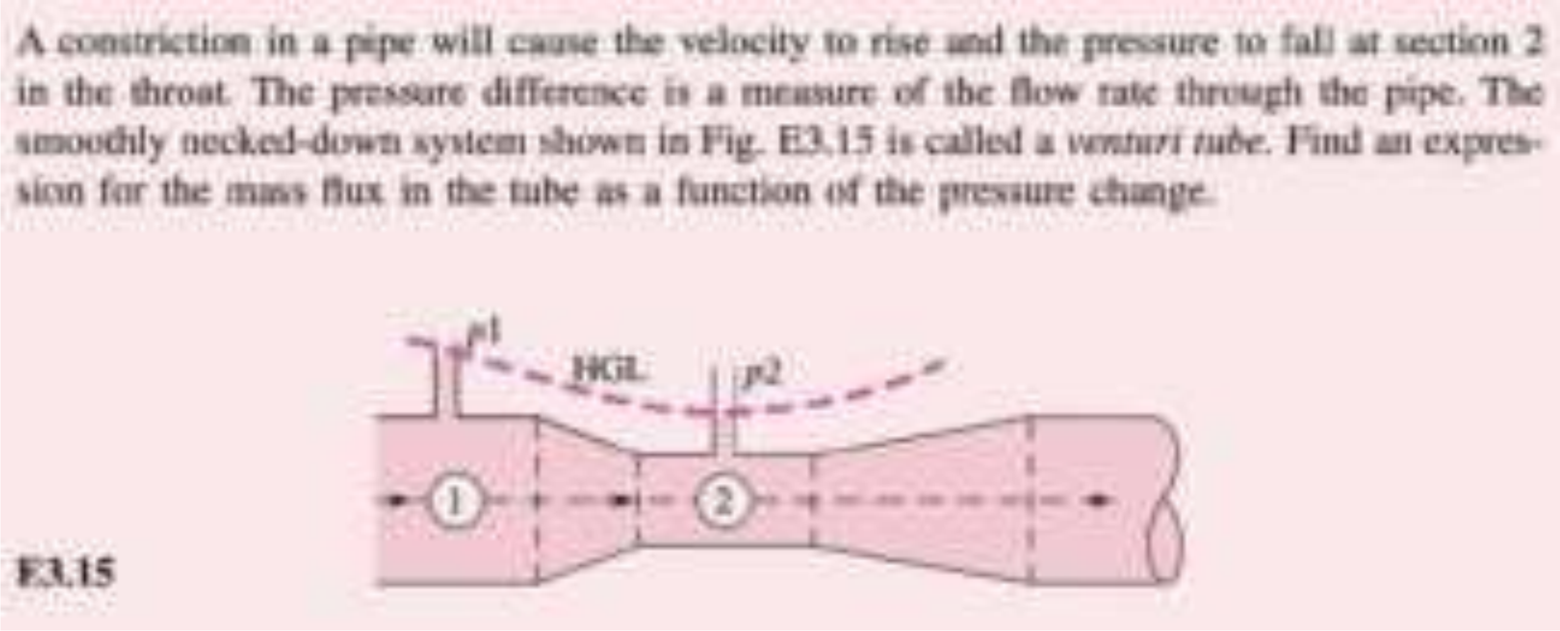
$$\frac{P_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{P_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$V_1 \approx 0$ and $P_2 = P_1 = P_{atm}$

$$gz_1 = \frac{V_2^2}{2} gz_2 \Rightarrow V_2 = \sqrt{2g(z_1 - z_2)}$$

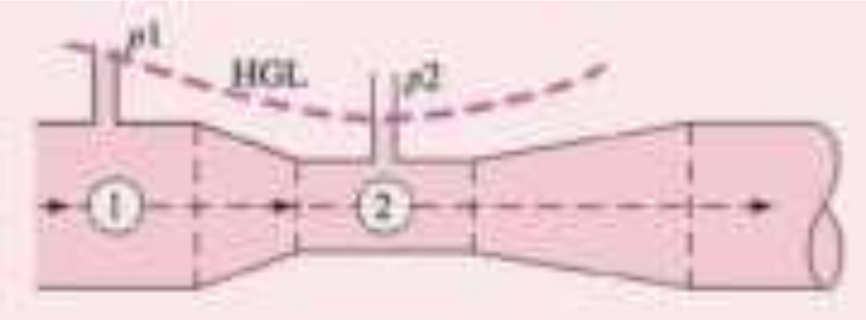
No siphon effect when $z_2 > z_1$.

Numerical values: $V_2 = 4.08\ m/s$ $Q = V_2 A_2 = 321\ cm^3/s$



A constriction in a pipe will cause the velocity to rise and the pressure to fall at section 2 in the throat. The pressure difference is a measure of the flow rate through the pipe. The smoothly necked-down system shown in Fig. E3.15 is called a *venturi tube*. Find an expression for the mass flux in the tube as a function of the pressure change.

E3.15



$$\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1 = \frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2$$

$$z_1 = z_2$$

$$V_2^2 - V_1^2 = \frac{2\Delta p}{\rho} \qquad \Delta p = p_1 - p_2$$

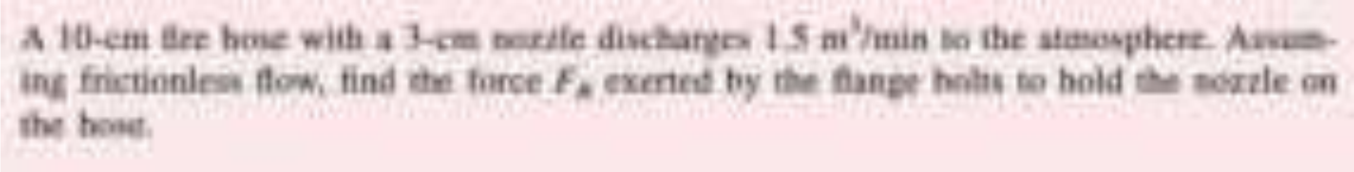
Incompressible continuity eqn.: $A_1 V_1 = A_2 V_2$

$$\frac{\pi}{4} D_1^2 V_1 = \frac{\pi}{4} D_2^2 V_2 \Rightarrow V_1 = \beta^2 V_2 \qquad \beta = \frac{D_2}{D_1}$$

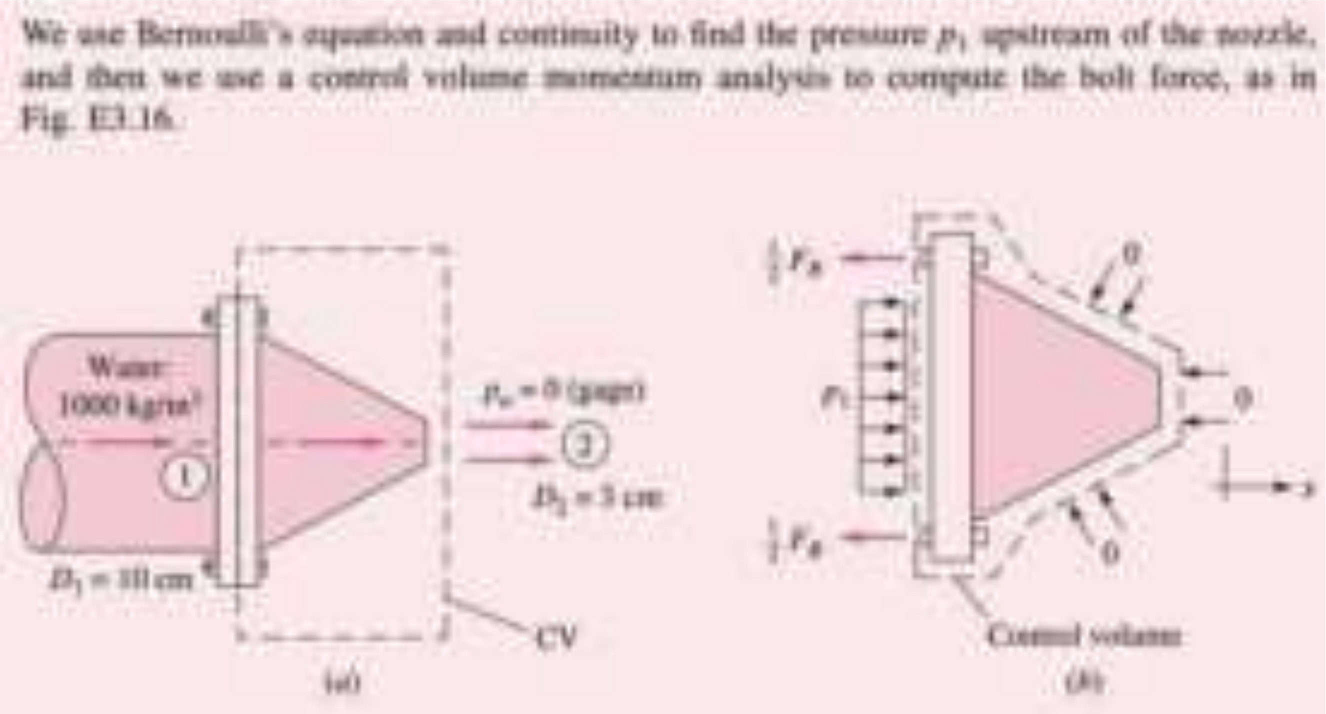
$$V_2^2(1-\beta^4) = \frac{2\Delta p}{\rho} \Rightarrow V_2 = \left[\frac{2\Delta p}{\rho(1-\beta^4)}\right]^{1/2}$$

$$\dot{m} = \rho A_2 V_2 = A_2\left(\frac{2\rho\Delta p}{1-\beta^4}\right)^{1/2}$$: frictionless mass flux.

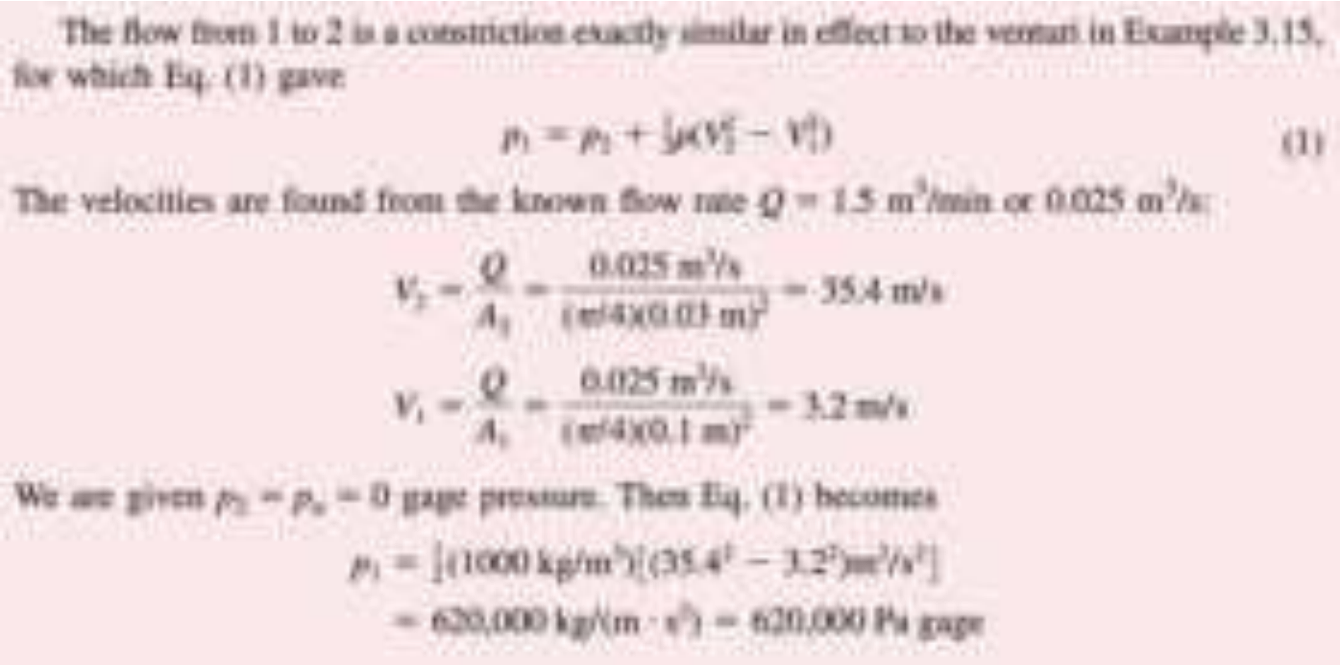
In practice, we measure $\dot{m}_{actual} = C_d \dot{m}_{ideal}$ and find the dimensionless discharge coef. C_d.



A 10-cm fire hose with a 3-cm nozzle discharges 1.5 m^3/min to the atmosphere. Assuming frictionless flow, find the force F_B exerted by the flange bolts to hold the nozzle on the hose.



We use Bernoulli's equation and continuity to find the pressure p_1 upstream of the nozzle, and then we use a control volume momentum analysis to compute the bolt force, as in Fig. E3.16.



The flow from 1 to 2 is a constriction exactly similar in effect to the venturi in Example 3.15, for which Eq. (1) gave

$$p_1 = p_2 + \tfrac{1}{2}\rho(V_2^2 - V_1^2) \qquad (1)$$

The velocities are found from the known flow rate $Q = 1.5\ \text{m}^3/\text{min}$ or $0.025\ \text{m}^3/\text{s}$:

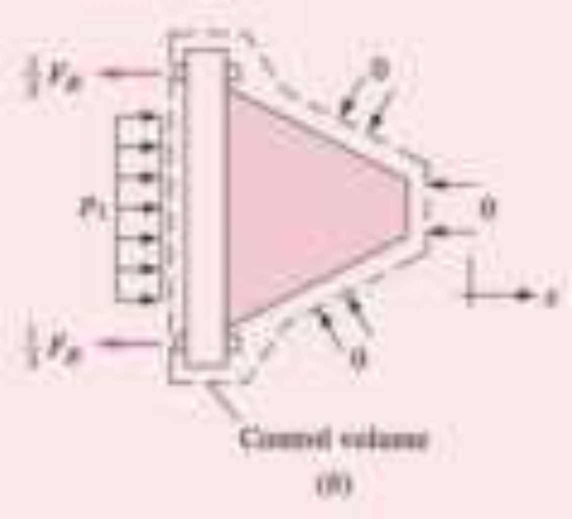
$$V_2 = \frac{Q}{A_2} = \frac{0.025\ \text{m}^3/\text{s}}{(\pi/4)(0.03\ \text{m})^2} = 35.4\ \text{m/s}$$

$$V_1 = \frac{Q}{A_1} = \frac{0.025\ \text{m}^3/\text{s}}{(\pi/4)(0.1\ \text{m})^2} = 3.2\ \text{m/s}$$

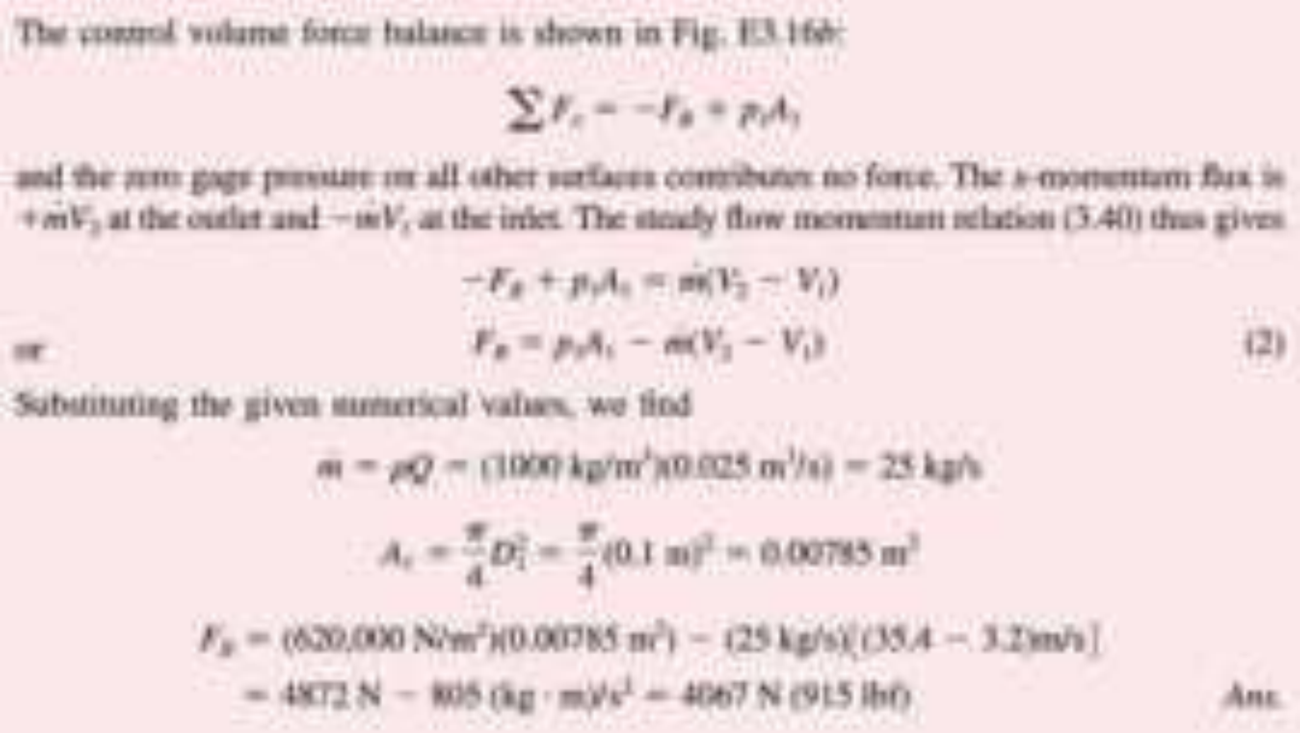
We are given $p_2 = p_a = 0$ gage pressure. Then Eq. (1) becomes

$$p_1 = \tfrac{1}{2}(1000\ \text{kg/m}^3)[(35.4^2 - 3.2^2)\text{m}^2/\text{s}^2]$$

$$= 620{,}000\ \text{kg/(m}\cdot\text{s}^2) = 620{,}000\ \text{Pa gage}$$



(b)



The control volume force balance is shown in Fig. E3.16b:

$$\sum F_x = -F_B + p_1 A_1$$

and the zero gage pressure on all other surfaces contributes no force. The x-momentum flux is $+\dot{m}V_2$ at the outlet and $-\dot{m}V_1$ at the inlet. The steady flow momentum relation (3.40) thus gives

$$-F_B + p_1 A_1 = \dot{m}(V_2 - V_1)$$

or

$$F_B = p_1 A_1 - \dot{m}(V_2 - V_1) \qquad (2)$$

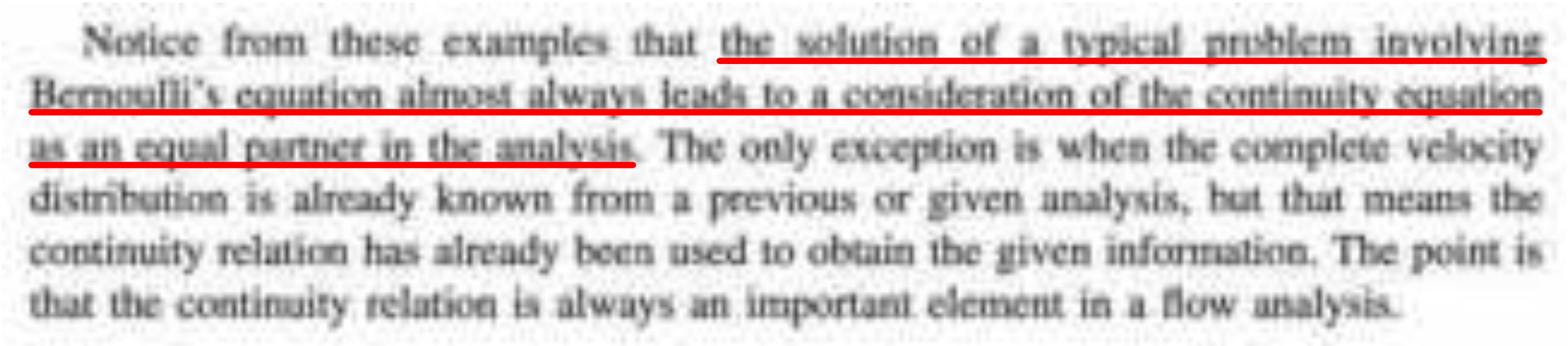
Substituting the given numerical values, we find

$$\dot{m} = \rho Q = (1000\ \text{kg/m}^3)(0.025\ \text{m}^3/\text{s}) = 25\ \text{kg/s}$$

$$A_1 = \frac{\pi}{4}D_1^2 = \frac{\pi}{4}(0.1\ \text{m})^2 = 0.00785\ \text{m}^2$$

$$F_B = (620{,}000\ \text{N/m}^2)(0.00785\ \text{m}^2) - (25\ \text{kg/s})[(35.4 - 3.2)\text{m/s}]$$

$$= 4872\ \text{N} - 805\ (\text{kg}\cdot\text{m})/\text{s}^2 = 4067\ \text{N}\ (915\ \text{lbf}) \qquad \textit{Ans.}$$



Notice from these examples that the solution of a typical problem involving Bernoulli's equation almost always leads to a consideration of the continuity equation as an equal partner in the analysis. The only exception is when the complete velocity distribution is already known from a previous or given analysis, but that means the continuity relation has already been used to obtain the given information. The point is that the continuity relation is always an important element in a flow analysis.

3.6) The Angular Momentum Theorem:

We omit this section. However, you should be wary of that it has direct applications to many important fluid flow problems.

A particularly important case is the analysis of rotating flow devices, usually called turbomachines.

3.7) The Energy Equation:

$$\frac{d}{dt}B_{sys} = \frac{d}{dt}\int_{CV}\beta\rho\, d\forall + \int_{CS}\beta\rho(\vec{V}\cdot\vec{n})\,dA$$

$B = E$ $\qquad \beta = \frac{dE}{dm} = e$ $\qquad$ 1st Law of thermodynamics : $\frac{dE}{dt} = \dot{Q} - \dot{W}$

$$\frac{dQ}{dt} - \frac{dW}{dt} = \frac{dE}{dt} = \frac{d}{dt}\int_{CV} e\rho\, d\forall + \int_{CS} e\rho(\vec{V}\cdot\vec{n})\,dA$$

e may be of several types:

$e = e_{internal} + e_{kinetic} + e_{potential} + e_{other}$

e_{other} could be chemical reactions, nuclear reactions, electrostatic & magnetic field effects. We neglect it.

$$e = \hat{u} + \frac{1}{2}\vec{v}^2 + gz$$

$\frac{dQ}{dt}$: could be conduction, convection, radiation effects

$\dot{W} = \dot{W}_{shaft} + \dot{W}_{press} + \dot{W}_{\underset{\text{stresses}}{viscous}} = \dot{W}_s + \dot{W}_p + \dot{W}_v$

$\dot{W}_p$: occurs at the surface only.

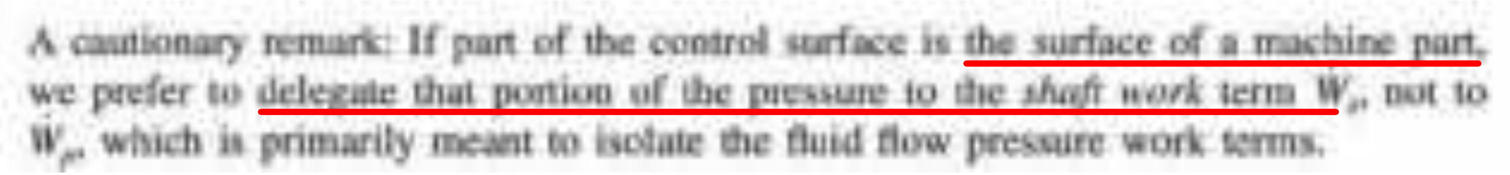
All work on the internal portions of the material in the CV is by equal & opposite forces and is self-canceling.

$$d\dot{W}_p = -(p\,dA)\,V_{n,in}$$

$$= -p\,(-\vec{V}\cdot\vec{n})\,dA$$

$V_{n,in}$: normal velocity into the CV

$$\dot{W}_p = \int_{cs} p\,(\vec{V}\cdot\vec{n})\,dA$$



A cautionary remark: If part of the control surface is the surface of a machine part, we prefer to delegate that portion of the pressure to the *shaft work* term W_s, not to W_p, which is primarily meant to isolate the fluid flow pressure work terms.

$$\dot{Q} - \dot{W}_s - \dot{W}_p - \dot{W}_v = \frac{d}{dt}\int_{CV} e\rho\, d\mathcal{V} + \int_{CS} e\rho(\vec{V}\cdot\vec{n})\, dA$$

$$\int_{CS} p(\vec{V}\cdot\vec{n})\, dA$$

$d\mathcal{V}$ is fixed

$$\dot{Q} - \dot{W}_s - \dot{W}_v = \frac{\partial}{\partial t}\int_{CV} e\rho\, d\mathcal{V} + \int_{CS}\left(e + \frac{p}{\rho}\right)\rho(\vec{V}\cdot\vec{n})\, dA$$

$$e = \hat{u} + \frac{1}{2}V^2 + gz \qquad h = \hat{u} + \frac{p}{\rho}$$

The energy eqn. for a fixed control volume:

$$\boxed{\begin{aligned}\dot{Q} - \dot{W}_s - \dot{W}_v &= \frac{\partial}{\partial t}\int_{CV}\left(\hat{u} + \frac{1}{2}V^2 + gz\right)\rho\, d\mathcal{V} \\ &\quad + \int_{CS}\left(\hat{h} + \frac{1}{2}V^2 + gz\right)\rho\,(\vec{V}\cdot\vec{n})\, dA\end{aligned}}$$

Note that the shear work term $\dot{W}_v$ is rarely important.

One-Dimensional Energy Flux Terms:

$$\int_{cs} \left(\hat{h} + \frac{1}{2}v^2 + gz\right) \rho \left(\vec{v} \cdot \vec{n}\right) dA$$

$$= \sum \left(\hat{h} + \frac{1}{2}v^2 + gz\right)_{out} \dot{m}_{out} - \sum \left(\hat{h} + \frac{1}{2}v^2 + gz\right)_{in} \dot{m}_{in}$$

$\hat{h}$, $\frac{1}{2}v^2$, gz: taken to be averages over each cross section

The Steady Flow Energy Equation:

For steady flow with one inlet & one outlet (1d)

$$\dot{Q} - \dot{W}_s - \dot{W}_v = -\dot{m}_1\left(\hat{h}_1 + \frac{1}{2}v_1^2 + gz_1\right) + \dot{m}_2\left(\hat{h}_2 + \frac{1}{2}v_1^2 + gz_2\right)$$

Continuity: $\dot{m}_1 = \dot{m}_2 = \dot{m}$

$$\hat{h}_1 + \frac{1}{2}v_1^2 + gz_1 = \hat{h}_2 + \frac{1}{2}v_2^2 + gz_2 - q + w_s + w_r$$

$q = \dfrac{\dot{Q}}{\dot{m}} = \dfrac{dQ}{dm}$: heat trans. per unit mass $\quad w_s = \dfrac{\dot{W}_s}{\dot{m}} = \dfrac{dW_s}{dm}$, $w_r = \dfrac{\dot{W}_r}{\dot{m}} = \dfrac{dW_r}{dm}$

This is a general form of the *steady flow energy equation.*

Divide by g: each term becomes a length, or head

$$\frac{p_1}{\gamma} + \frac{\hat{u}_1}{g} + \frac{v_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\hat{u}_2}{g} + \frac{v_2^2}{2g} + z_2 - h_q + h_s + h_r$$

$\frac{p}{\gamma}$: pressure head

$\frac{v^2}{2g}$: velocity head

$h_q = \frac{q}{g}$

$h_s = \frac{w_s}{g}$

$h_r = \frac{w_r}{g}$

h_q, h_s, h_r: head forms of the heat added, shaft work done, viscous work done

Friction & Shaft Work in Low-speed Flow:

Low speed (incompressible) flow through a pipe or duct:

A pump or turbine may be included in the pipe system.

The pipe & machine walls are solid ⇒ viscous work = 0

$$\underbrace{\underbrace{\left(\frac{P_1}{\gamma}+\frac{V_1^2}{2g}+z_1\right)}=\underbrace{\left(\frac{P_2}{\gamma}+\frac{V_2^2}{2g}+z_2\right)}}_{\text{upstream \& downstream values of the available head or total head } \equiv h_0}+\underbrace{\frac{\hat{u}_2-\hat{u}_1-q}{g}}_{h_{01}-h_{02}}$$

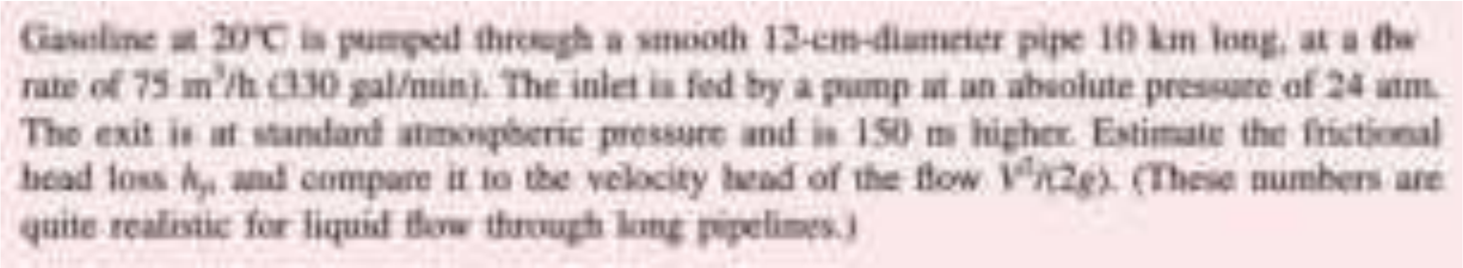
$h_{01} - h_{02}$: can include pump head input, turbine head extraction, friction head loss

In incompressible flow with one inlet & one outlet:

$$\boxed{\left(\frac{P}{\gamma} + \frac{V^2}{2g} + z\right)_{in} = \left(\frac{P}{\gamma} + \frac{V^2}{2g} + z\right)_{out} + h_{friction} - h_{pump} + h_{turbine}}$$

All the h terms are positive!

If hpump, hturbine are included, they must lie between 1&2.



Gasoline at 20°C is pumped through a smooth 12-cm-diameter pipe 10 km long, at a flow rate of 75 m^3/h (330 gal/min). The inlet is fed by a pump at an absolute pressure of 24 atm. The exit is at standard atmospheric pressure and is 150 m higher. Estimate the frictional head loss h_f, and compare it to the velocity head of the flow $V^2/(2g)$. (These numbers are quite realistic for liquid flow through long pipelines.)

$$V_{in} = V_{out} = \frac{Q}{A} = \frac{Q}{(\pi/4)D^2} = \frac{(75\ \text{m}^3/\text{h})/(3600\ \text{s/h})}{(\pi/4)(0.12\ \text{m})^2} \approx 1.84\ \frac{\text{m}}{\text{s}}$$

$$\text{Velocity head} = \frac{V^2}{2g} = \frac{(1.84\ \text{m/s})^2}{2(9.81\ \text{m/s}^2)} \approx 0.173\ \text{m}$$

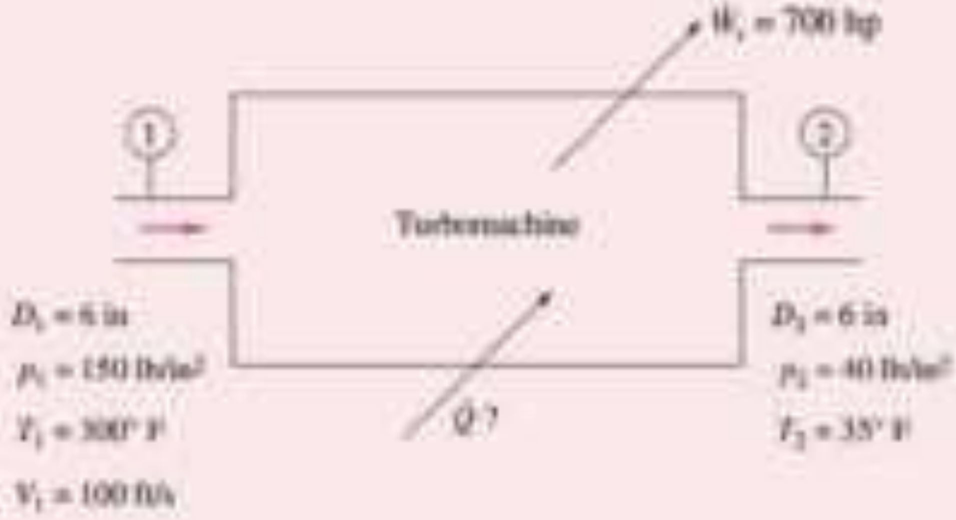
$$\frac{p_{in}}{\gamma} + \frac{V_{in}^2}{2g} + z_{in} = \frac{p_{out}}{\gamma} + \frac{V_{out}^2}{2g} + z_{out} + h_f$$

$$\frac{(24)(101{,}350\ \text{N/m}^2)}{6670\ \text{N/m}^3} + 0.173\ \text{m} + 0\ \text{m} = \frac{101{,}350\ \text{N/m}^2}{6670\ \text{N/m}^3} + 0.173\ \text{m} + 150\ \text{m} + h_f$$

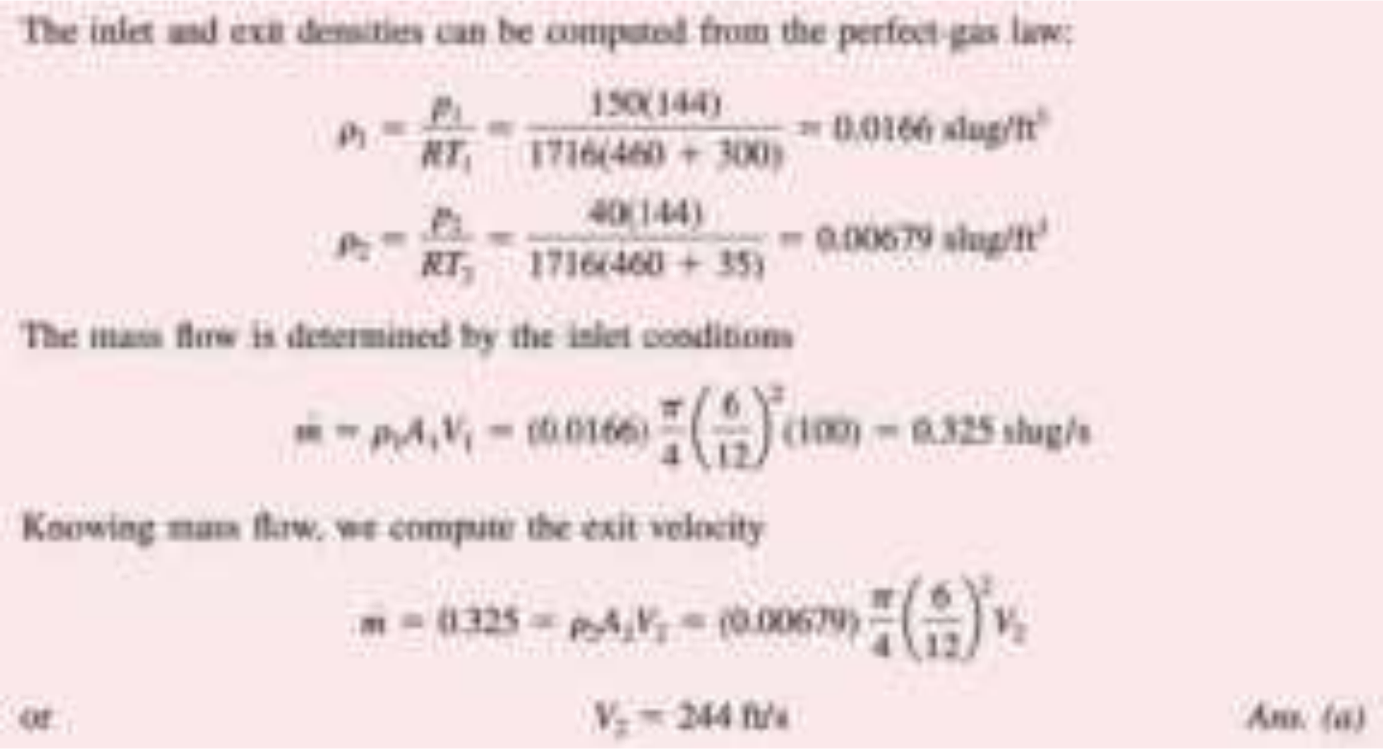
or $$h_f = 364.7 - 15.2 - 150 \approx 199\ \text{m}$$ *Ans.*

$$\frac{h_f}{V^2/(2g)} = \frac{199\ \text{m}}{0.173\ \text{m}} \approx 1150$$

Air [R = 1716, c_p = 6003 ft · lbf/(slug · °R)] flows steadily, as shown in Fig. E3.22, through a turbine that produces 700 hp. For the inlet and exit conditions shown, estimate (*a*) the exit velocity V_2 and (*b*) the heat transferred Q in Btu/h.



E3.22



The inlet and exit densities can be computed from the perfect-gas law:

$$\rho_1 = \frac{p_1}{RT_1} = \frac{150(144)}{1716(460 + 300)} = 0.0166 \text{ slug/ft}^3$$

$$\rho_2 = \frac{p_2}{RT_2} = \frac{40(144)}{1716(460 + 35)} = 0.00679 \text{ slug/ft}^3$$

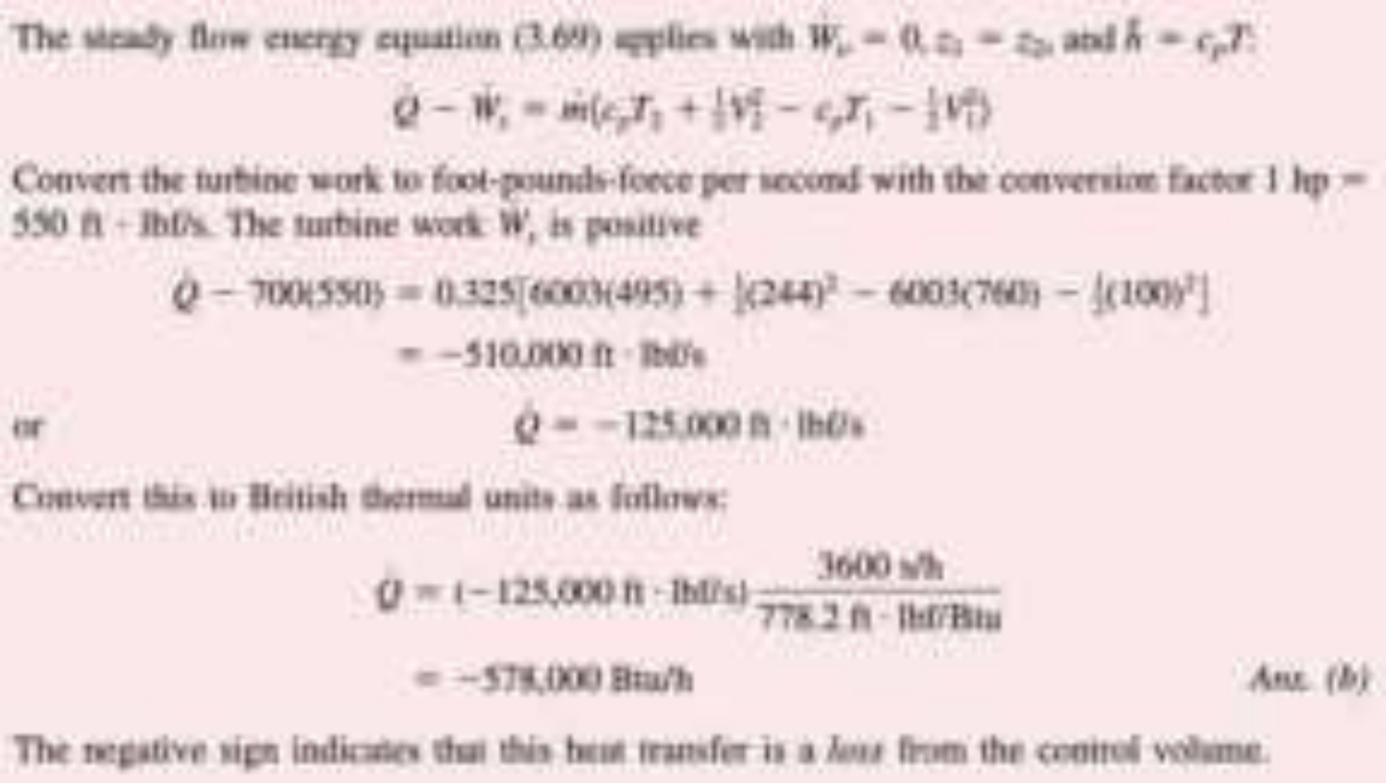
The mass flow is determined by the inlet conditions

$$\dot{m} = \rho_1 A_1 V_1 = (0.0166)\,\frac{\pi}{4}\left(\frac{6}{12}\right)^2 (100) = 0.325 \text{ slug/s}$$

Knowing mass flow, we compute the exit velocity

$$\dot{m} = 0.325 = \rho_2 A_2 V_2 = (0.00679)\,\frac{\pi}{4}\left(\frac{6}{12}\right)^2 V_2$$

or $V_2 = 244$ ft/s *Ans. (a)*



The steady flow energy equation (3.69) applies with $\dot{W}_v = 0$, $z_1 = z_2$, and $\hat{h} = c_p T$:

$$\dot{Q} - \dot{W}_s = \dot{m}\left(c_p T_2 + \tfrac{1}{2}V_2^2 - c_p T_1 - \tfrac{1}{2}V_1^2\right)$$

Convert the turbine work to foot-pounds-force per second with the conversion factor 1 hp = 550 ft · lbf/s. The turbine work $\dot{W}_s$ is positive

$$\dot{Q} - 700(550) = 0.325\left[6003(495) + \tfrac{1}{2}(244)^2 - 6003(760) - \tfrac{1}{2}(100)^2\right]$$

$$= -510{,}000 \text{ ft} \cdot \text{lbf/s}$$

or

$$\dot{Q} = -125{,}000 \text{ ft} \cdot \text{lbf/s}$$

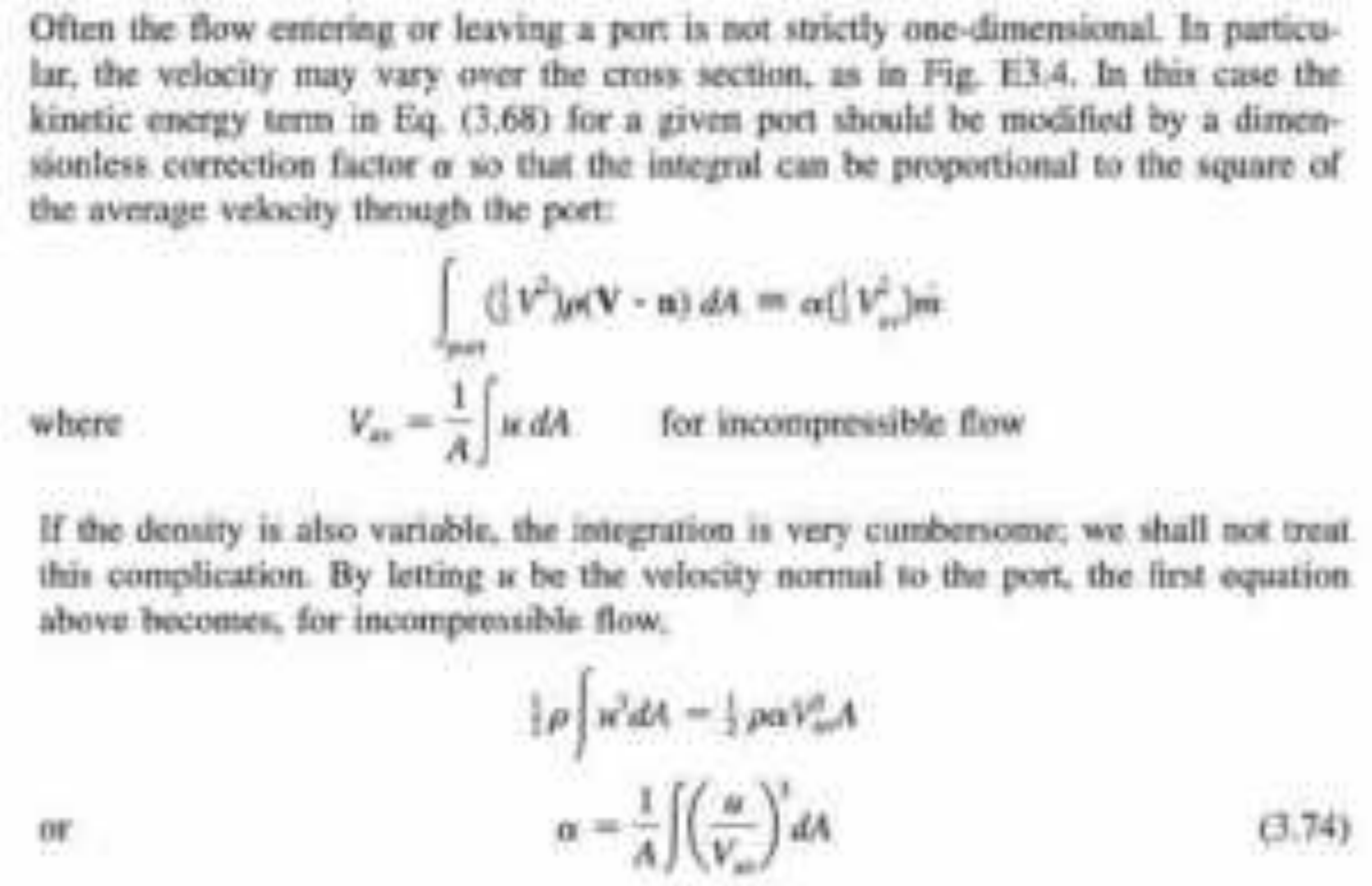
Convert this to British thermal units as follows:

$$\dot{Q} = (-125{,}000 \text{ ft} \cdot \text{lbf/s}) \frac{3600 \text{ s/h}}{778.2 \text{ ft} \cdot \text{lbf/Btu}}$$

$$= -578{,}000 \text{ Btu/h} \qquad \textit{Ans. (b)}$$

The negative sign indicates that this heat transfer is a *loss* from the control volume.

Kinetic Energy Correction Factor:



Often the flow entering or leaving a port is not strictly one-dimensional. In particular, the velocity may vary over the cross section, as in Fig. E3.4. In this case the kinetic energy term in Eq. (3.68) for a given port should be modified by a dimensionless correction factor α so that the integral can be proportional to the square of the average velocity through the port:

$$\int_{\text{port}} \left(\tfrac{1}{2}V^2\right)\rho(\mathbf{V}\cdot\mathbf{n})\,dA \equiv \alpha\left(\tfrac{1}{2}V_{av}^2\right)\dot{m}$$

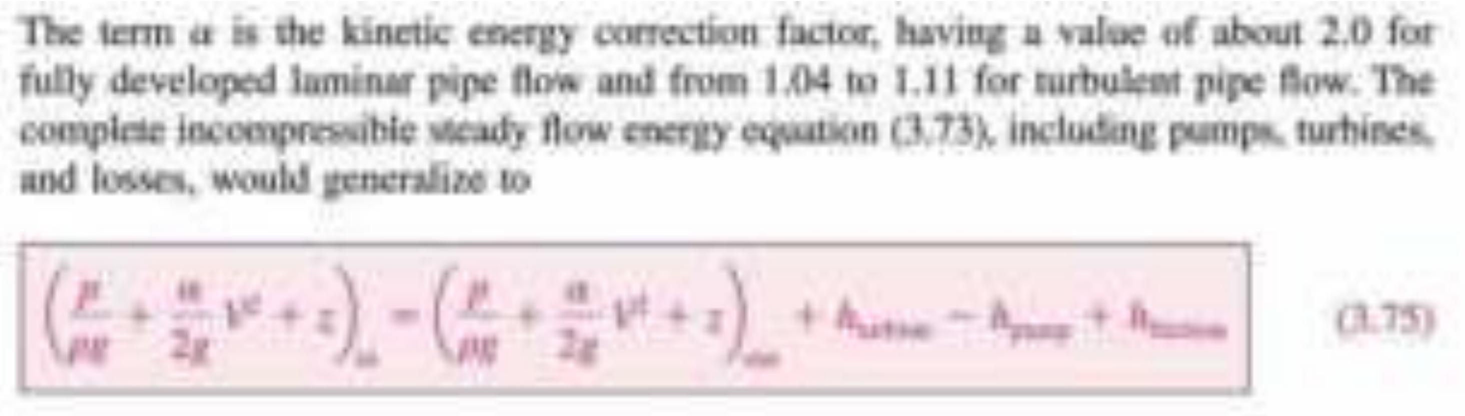
where $V_{av} = \dfrac{1}{A}\displaystyle\int u\,dA$ for incompressible flow

If the density is also variable, the integration is very cumbersome; we shall not treat this complication. By letting u be the velocity normal to the port, the first equation above becomes, for incompressible flow,

$$\tfrac{1}{2}\rho\int u^3\,dA = \tfrac{1}{2}\rho\alpha V_{av}^3 A$$

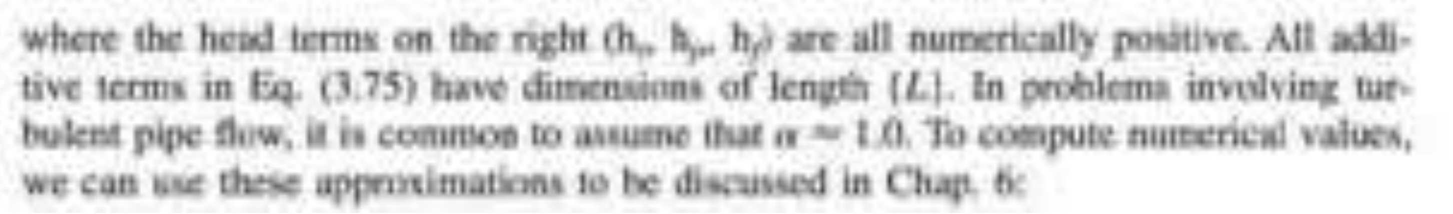
or

$$\alpha = \frac{1}{A}\int\left(\frac{u}{V_{av}}\right)^3 dA \qquad (3.74)$$



The term α is the kinetic energy correction factor, having a value of about 2.0 for fully developed laminar pipe flow and from 1.04 to 1.11 for turbulent pipe flow. The complete incompressible steady flow energy equation (3.73), including pumps, turbines, and losses, would generalize to

$$\left(\frac{p}{\rho g}+\frac{\alpha}{2g}V^2+z\right)_{in}=\left(\frac{p}{\rho g}+\frac{\alpha}{2g}V^2+z\right)_{out}+h_{turbine}-h_{pump}+h_{friction} \qquad (3.75)$$



where the head terms on the right (h_t, h_p, h_f) are all numerically positive. All additive terms in Eq. (3.75) have dimensions of length $\{L\}$. In problems involving turbulent pipe flow, it is common to assume that $\alpha \approx 1.0$. To compute numerical values, we can use these approximations to be discussed in Chap. 6:

Laminar flow:

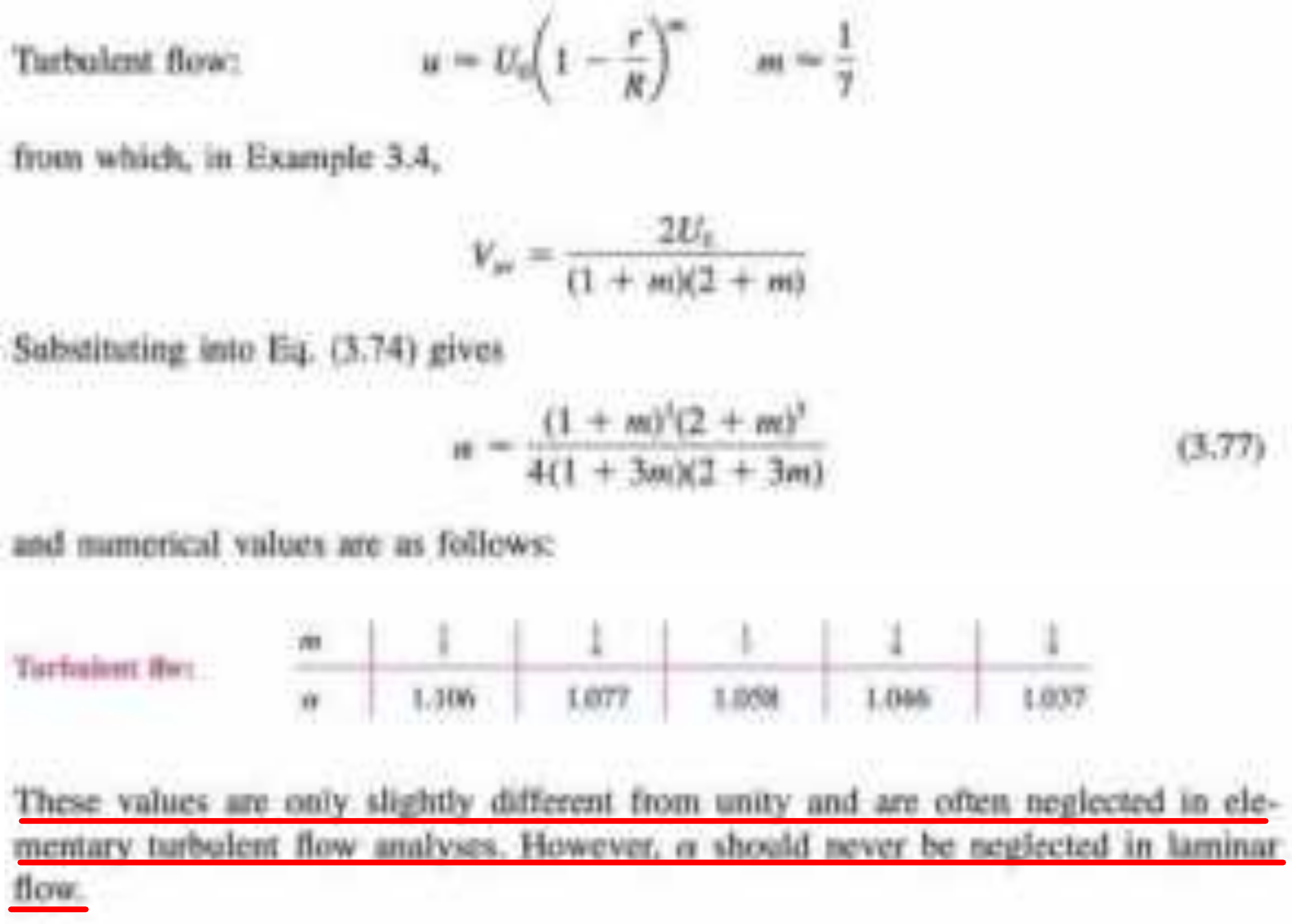
$$u = U_0\left[1 - \left(\frac{r}{R}\right)^2\right]$$

from which

$$V_{av} = 0.5U_0$$

and

$$\alpha = 2.0 \tag{3.76}$$



Turbulent flow:
$$u \approx U_0\left(1 - \frac{r}{R}\right)^m \qquad m \approx \frac{1}{7}$$

from which, in Example 3.4,

$$V_{av} = \frac{2U_0}{(1+m)(2+m)}$$

Substituting into Eq. (3.74) gives

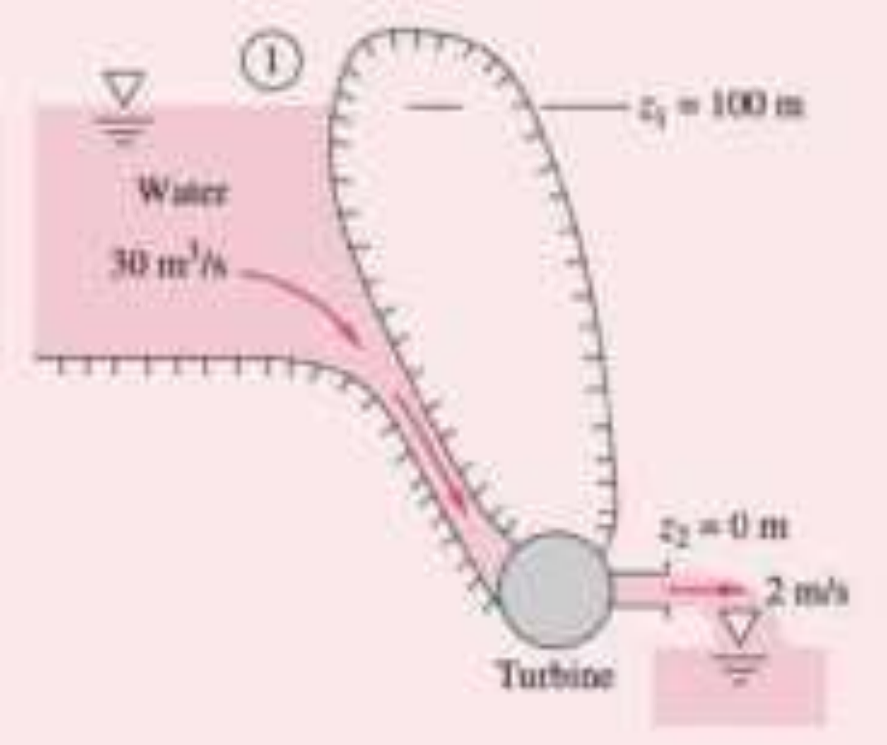
$$\alpha = \frac{(1+m)^3(2+m)^3}{4(1+3m)(2+3m)} \tag{3.77}$$

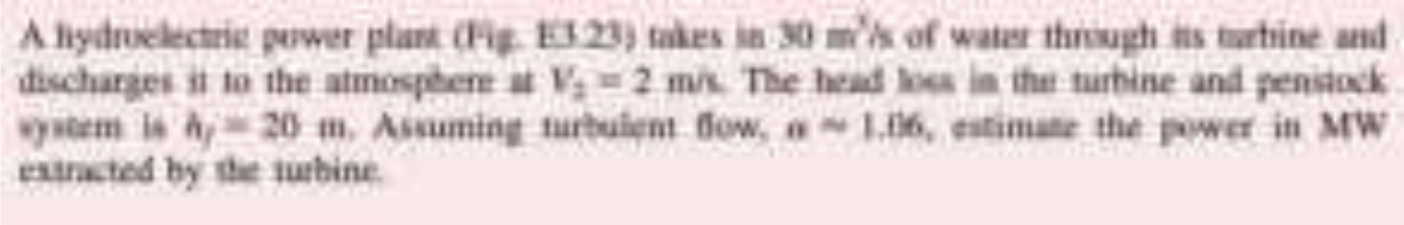
and numerical values are as follows:

Turbulent flow:

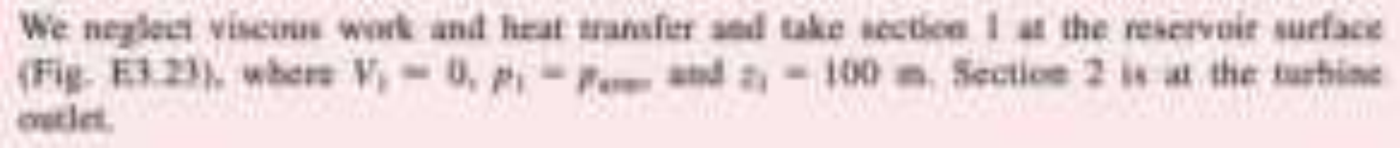
m	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{8}$	$\frac{1}{9}$
α	1.106	1.077	1.058	1.046	1.037

These values are only slightly different from unity and are often neglected in elementary turbulent flow analyses. However, α should never be neglected in laminar flow.

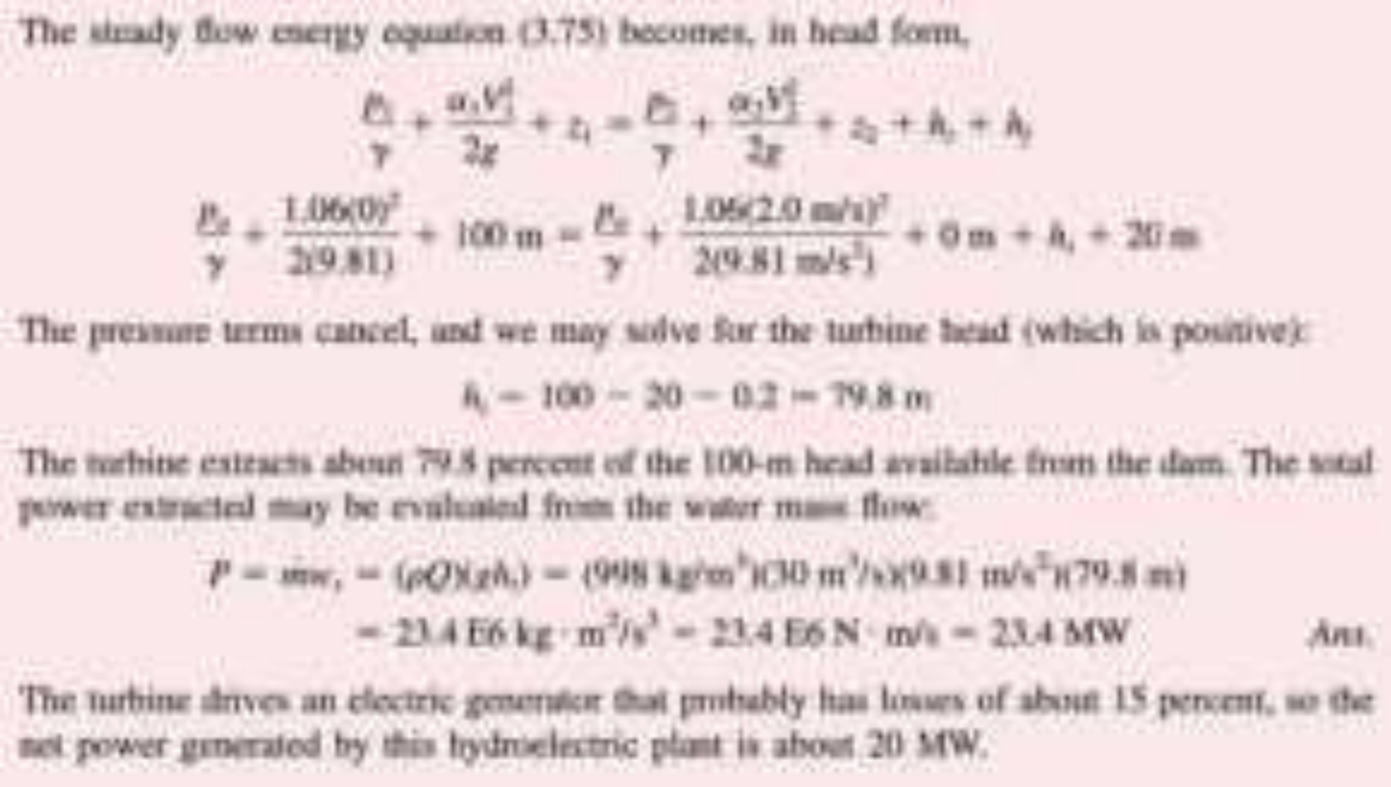




A hydroelectric power plant (Fig. E3.23) takes in 30 m^3/s of water through its turbine and discharges it to the atmosphere at V_2 = 2 m/s. The head loss in the turbine and penstock system is h_f = 20 m. Assuming turbulent flow, $\alpha \approx 1.06$, estimate the power in MW extracted by the turbine.



We neglect viscous work and heat transfer and take section 1 at the reservoir surface (Fig. E3.23), where $V_1 = 0$, $p_1 = p_{atm}$, and $z_1 = 100$ m. Section 2 is at the turbine outlet.



The steady flow energy equation (3.75) becomes, in head form,

$$\frac{p_1}{\gamma} + \frac{\alpha_1 V_1^2}{2g} + z_1 = \frac{p_2}{\gamma} + \frac{\alpha_2 V_2^2}{2g} + z_2 + h_t + h_f$$

$$\frac{p_a}{\gamma} + \frac{1.06(0)^2}{2(9.81)} + 100\text{ m} = \frac{p_a}{\gamma} + \frac{1.06(2.0\text{ m/s})^2}{2(9.81\text{ m/s}^2)} + 0\text{ m} + h_t + 20\text{ m}$$

The pressure terms cancel, and we may solve for the turbine head (which is positive):

$$h_t = 100 - 20 - 0.2 = 79.8\text{ m}$$

The turbine extracts about 79.8 percent of the 100-m head available from the dam. The total power extracted may be evaluated from the water mass flow:

$$P = \dot{m}w_t = (\rho Q)(g h_t) = (998\text{ kg/m}^3)(30\text{ m}^3/\text{s})(9.81\text{ m/s}^2)(79.8\text{ m})$$

$$= 23.4\text{ E6 kg}\cdot\text{m}^2/\text{s}^3 = 23.4\text{ E6 N}\cdot\text{m/s} = 23.4\text{ MW} \qquad \textit{Ans.}$$

The turbine drives an electric generator that probably has losses of about 15 percent, so the net power generated by this hydroelectric plant is about 20 MW.

4) DIFFERENTIAL RELATIONS FOR FLUID FLOW

In this chapter, we focus on the small scale or differential analysis.

We apply four basic conservation laws to an infinitesimal control volume, or, to an infinitesimal fluid system

The results yield the basic differential equations of fluid motion.

We will also develop the appropriate boundary conditions.

4.1) The Acceleration Field of a Fluid:

$$\vec{V}(\vec{r},t) = \hat{i}\,u(x,y,z,t) + \hat{j}\,v(x,y,z,t) + \hat{k}\,w(x,y,z,t)$$

The velocity field is the most important variable in fluid mechanics.

Solving a fluid flow problem requires determination of the velocity field.

Our coordinates are fixed in space: We observe the fluid as it passes by ⇒ Eulerian frame of reference

Lagrangian frame: It follows the moving position of individual particles.

$$\vec{a} = \frac{d\vec{V}}{dt} = \hat{\imath}\,\frac{du}{dt} + \hat{\jmath}\,\frac{dv}{dt} + \hat{k}\,\frac{dw}{dt}$$: total time derivative of the velocity field

$$\frac{d}{dt}\,u(x,y,z,t) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Remember $u = \frac{dx}{dt}$ $v = \frac{dy}{dt}$ $w = \frac{dz}{dt}$

$$\vec{V} = \frac{d\vec{r}}{dt} \qquad \vec{r} = \hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z$$

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + (\vec{V}\cdot\vec{\nabla})u$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + (\vec{V}\cdot\vec{\nabla})v$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + (\vec{V}\cdot\vec{\nabla})w$$

$$\Rightarrow \quad \vec{a} = \frac{d\vec{V}}{dt} = \underbrace{\frac{\partial \vec{V}}{\partial t}}_{\text{local}} + \underbrace{\left(u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z}\right)}_{\text{convective}}$$

$$\boxed{\begin{aligned} \vec{a} = \frac{d\vec{V}}{dt} &= \frac{\partial \vec{V}}{\partial t} + \left(u\frac{\partial \vec{V}}{\partial x} + v\frac{\partial \vec{V}}{\partial y} + w\frac{\partial \vec{V}}{\partial z} \right) \\ &= \frac{\partial \vec{V}}{\partial t} + (\vec{V}\cdot\vec{\nabla})\vec{V} \end{aligned}}$$

$\frac{\partial \vec{V}}{\partial t}$: local acceleration, vanishes when the flow is steady

$(\vec{V}\cdot\vec{\nabla})\vec{V}$: convective acceleration, arises when the particle moves through regions of spatially varying velocity

* Steady flows may have large accelerations due to the convective terms.

$$\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w \qquad \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{V}\cdot\vec{\nabla} = \left(\hat{i}u + \hat{j}v + \hat{k}w\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

The total time derivative, sometimes called the substantial or material derivative, concept may be applied to any variable:

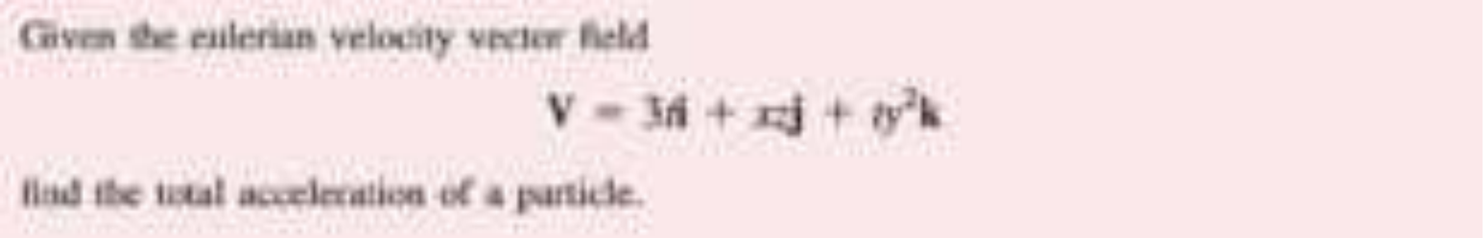
$$\frac{d\bullet}{dt} = \frac{\partial \bullet}{\partial t} + u\frac{\partial \bullet}{\partial x} + v\frac{\partial \bullet}{\partial y} + w\frac{\partial \bullet}{\partial z} = \frac{\partial \bullet}{\partial t} + (\vec{V}\cdot\vec{\nabla})\bullet$$

$\bullet$: any fluid variable

Ex: $$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + (\vec{V}\cdot\vec{\nabla})\rho$$

Due to convective effects in the basic laws, the differential equations become nonlinear & more complicated.

Sometimes the symbol $\frac{D}{dt}$ is used for this substantial or material derivative.

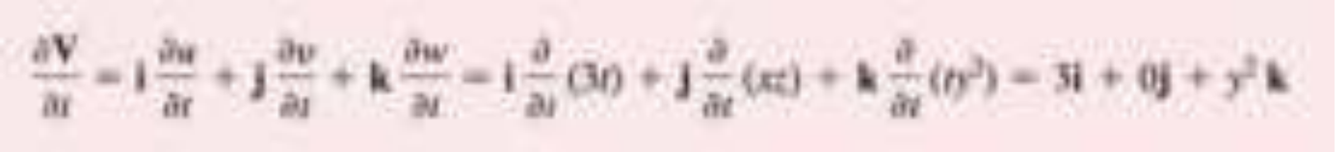


Given the eulerian velocity vector field

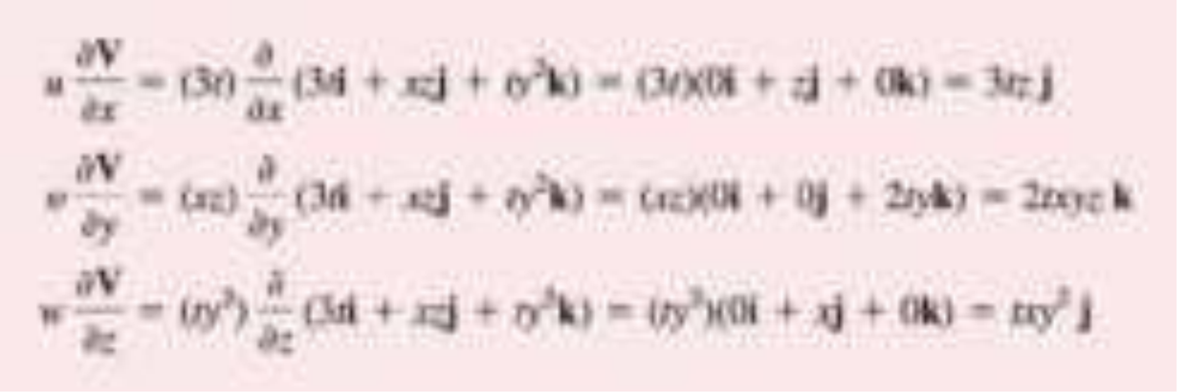
$$\mathbf{V} = 3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}$$

find the total acceleration of a particle.

$u = 3t \qquad v = xz \qquad w = ty^2$



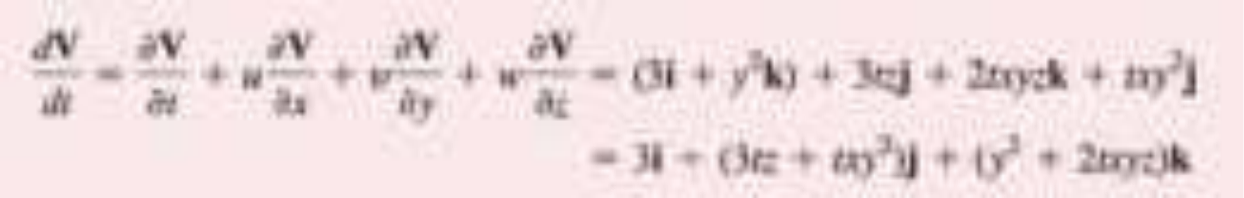
$$\frac{\partial \mathbf{V}}{\partial t} = \mathbf{i}\frac{\partial u}{\partial t} + \mathbf{j}\frac{\partial v}{\partial t} + \mathbf{k}\frac{\partial w}{\partial t} = \mathbf{i}\frac{\partial}{\partial t}(3t) + \mathbf{j}\frac{\partial}{\partial t}(xz) + \mathbf{k}\frac{\partial}{\partial t}(ty^2) = 3\mathbf{i} + 0\mathbf{j} + y^2\mathbf{k}$$



$$u\frac{\partial \mathbf{V}}{\partial x} = (3t)\frac{\partial}{\partial x}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (3t)(0\mathbf{i} + z\mathbf{j} + 0\mathbf{k}) = 3tz\,\mathbf{j}$$

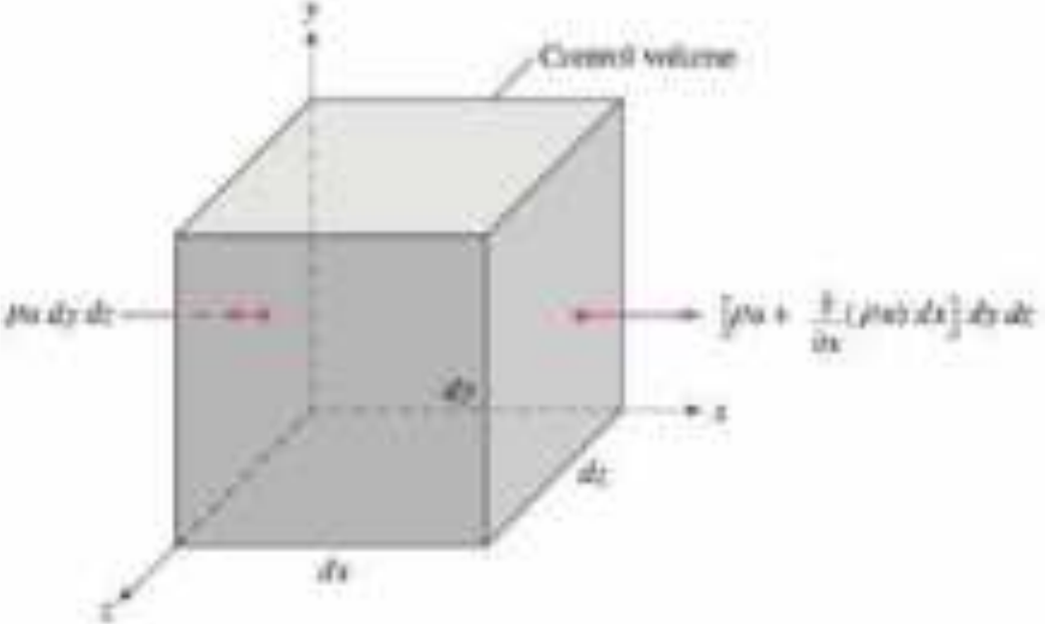
$$v\frac{\partial \mathbf{V}}{\partial y} = (xz)\frac{\partial}{\partial y}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (xz)(0\mathbf{i} + 0\mathbf{j} + 2ty\mathbf{k}) = 2txyz\,\mathbf{k}$$

$$w\frac{\partial \mathbf{V}}{\partial z} = (ty^2)\frac{\partial}{\partial z}(3t\mathbf{i} + xz\mathbf{j} + ty^2\mathbf{k}) = (ty^2)(0\mathbf{i} + x\mathbf{j} + 0\mathbf{k}) = txy^2\,\mathbf{j}$$



$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = (3\mathbf{i} + y^2\mathbf{k}) + 3tz\mathbf{j} + 2txyz\mathbf{k} + txy^2\mathbf{j}$$
$$= 3\mathbf{i} + (3tz + txy^2)\mathbf{j} + (y^2 + 2txyz)\mathbf{k}$$

4.2) The Differential Equation for Mass Conservation:



Conservation of mass is often called the continuity relation

All the basic differential equations can be derived by considering an elemental control volume (dx, dy, dz) as in the figure.

Mass Conservation : $\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho A_i V_i)_{in} = 0$

$$\int_{CV} \frac{\partial \rho}{\partial t} d\mathcal{V} \approx \frac{\partial \rho}{\partial t} dx\,dy\,dz$$

The mass flow terms occur on all six faces: three inlets & three outlets.

We make use of the field or continuum concept: All fluid properties are uniformly varying functions of time & position.

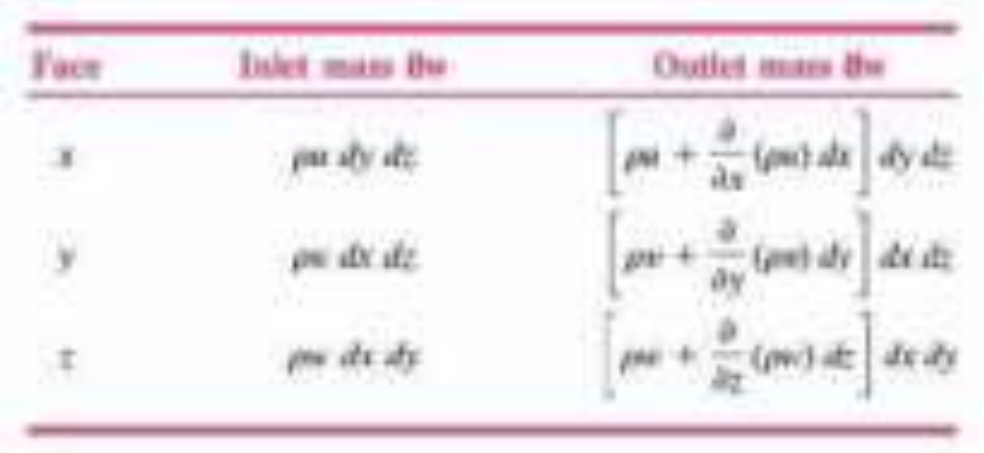
Ex: Left face has temp. T

Right face has temp $T + \frac{\partial T}{\partial x} dx$

why? $T(x+dx, y, z, t) = T(x, y, z, t) + \frac{\partial T}{\partial x} dx$

Similarly, $\rho u \big|_{\substack{\text{right}\\\text{face}}} = \rho u + \frac{\partial}{\partial x}(\rho u)$

We can list all six flows as follows:



Face	Inlet mass flow	Outlet mass flow
x	$\rho u \, dy \, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u) \, dx\right] dy \, dz$
y	$\rho v \, dx \, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v) \, dy\right] dx \, dz$
z	$\rho w \, dx \, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w) \, dz\right] dx \, dy$

$$
\begin{aligned}
0 = \frac{\partial \rho}{\partial t}\,dx\,dy\,dz &+ \left[\overset{\swarrow}{\rho u} + \frac{\partial}{\partial x}(\rho u)\,dx\right] dy\,dz - \overset{\swarrow}{\rho u}\,dy\,dz \\
&+ \left[\overset{\swarrow}{\rho v} + \frac{\partial}{\partial y}(\rho v)\,dy\right] dx\,dz - \overset{\swarrow}{\rho v}\,dx\,dz \\
&+ \left[\overset{\swarrow}{\rho w} + \frac{\partial}{\partial z}(\rho w)\,dz\right] dx\,dy - \overset{\swarrow}{\rho w}\,dx\,dy
\end{aligned}
$$

Terms with colored arrows cancel each other.

The volume element cancels out of all terms

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0}$$

This is the equation for conservation of mass for an infinitesimal control volume.

It is often called the equation of continuity.

Note that the equation does not allow for any source or sink singularities within the element.

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad \rho\vec{V} = \hat{i}\rho u + \hat{j}\rho v + \hat{k}\rho w$$

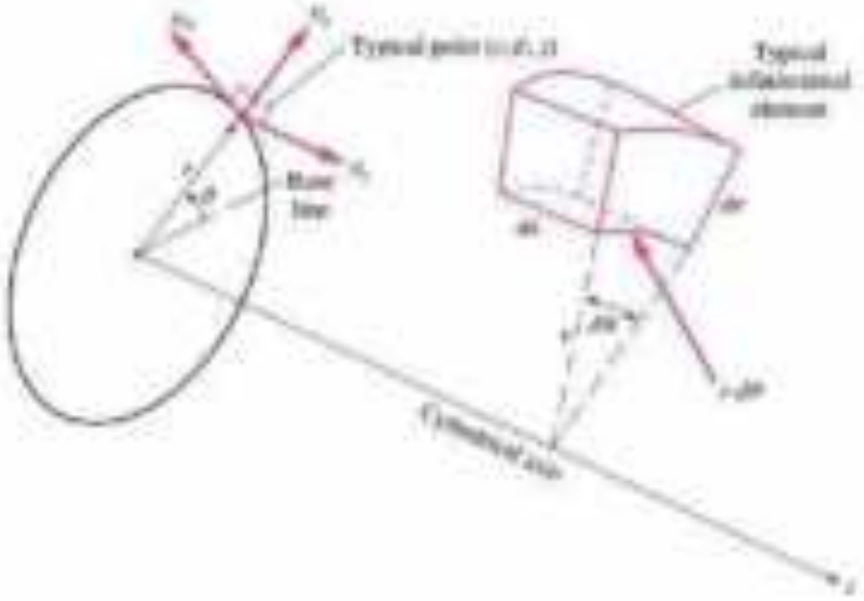
$$\vec{\nabla}\cdot(\rho\vec{V}) = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)$$

The compact form of the continuity relation:

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0}$$

This form is quite general & can be easily converted to other coordinate systems.

Cylindrical Polar Coordinate System:



$\vec{A} = \vec{A}(r, \theta, z, t)$: arbitrary vector function

Its divergence can be found by making the transformation of coords:

$r = (x^2 + y^2)^{1/2}$ $\theta = \tan^{-1}(y/x)$

$z = z$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$ (You don't have to know the proof.)

The continuity eqn. in cylindirical polar coords:

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0}$$

Steady Compressible Flow: $\frac{\partial}{\partial t} \equiv 0$

Cartesian: $$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical: $$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

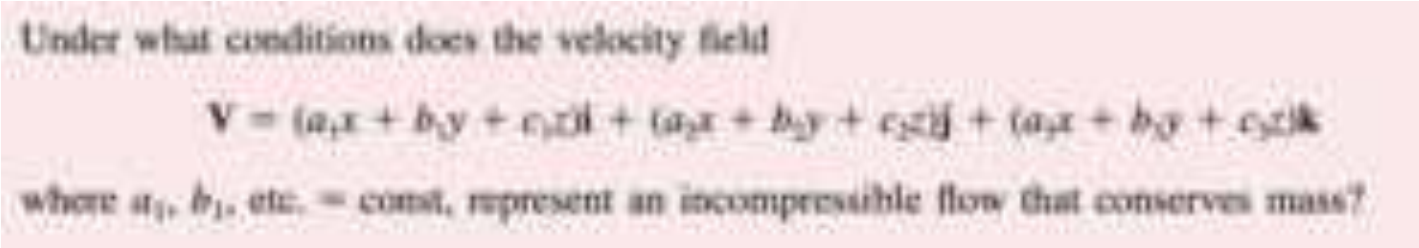
Incompressible Flow: $\rho = \text{const}$ $\quad \frac{\partial \rho}{\partial t} = 0$

$$\vec{\nabla} \cdot (\rho \vec{v}) = 0 \quad \Rightarrow \quad \vec{\nabla} \cdot \vec{v} = 0$$

Cartesian: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Cylindrical: $\frac{1}{r} \frac{\partial}{\partial r}(r v_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$

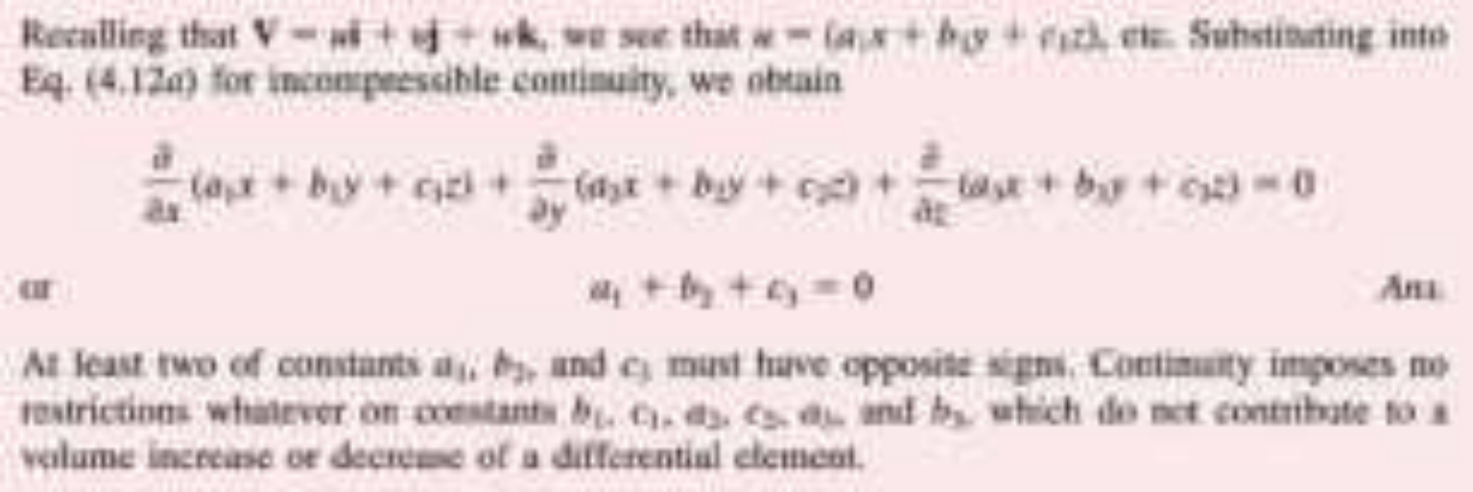
These are linear differential equations and a wide variety of solutions are known.



Under what conditions does the velocity field

$$\mathbf{V} = (a_1x + b_1y + c_1z)\mathbf{i} + (a_2x + b_2y + c_2z)\mathbf{j} + (a_3x + b_3y + c_3z)\mathbf{k}$$

where a_1, b_1, etc. = const, represent an incompressible flow that conserves mass?



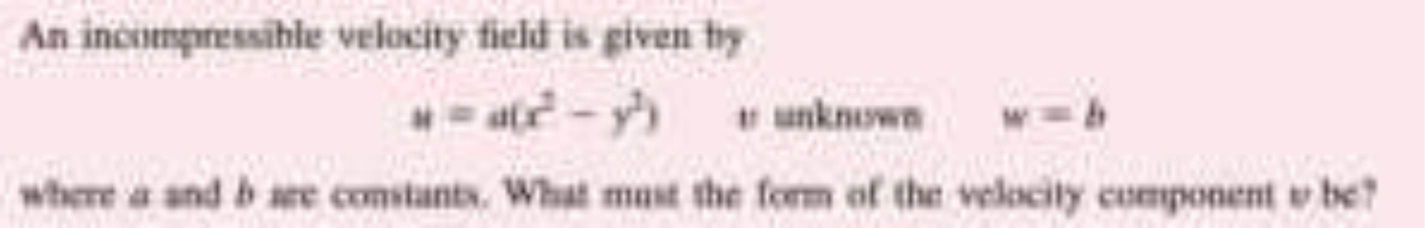
Recalling that $\mathbf{V} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, we see that $u = (a_1x + b_1y + c_1z)$, etc. Substituting into Eq. (4.12*a*) for incompressible continuity, we obtain

$$\frac{\partial}{\partial x}(a_1x + b_1y + c_1z) + \frac{\partial}{\partial y}(a_2x + b_2y + c_2z) + \frac{\partial}{\partial z}(a_3x + b_3y + c_3z) = 0$$

or

$$a_1 + b_2 + c_3 = 0 \qquad \text{Ans.}$$

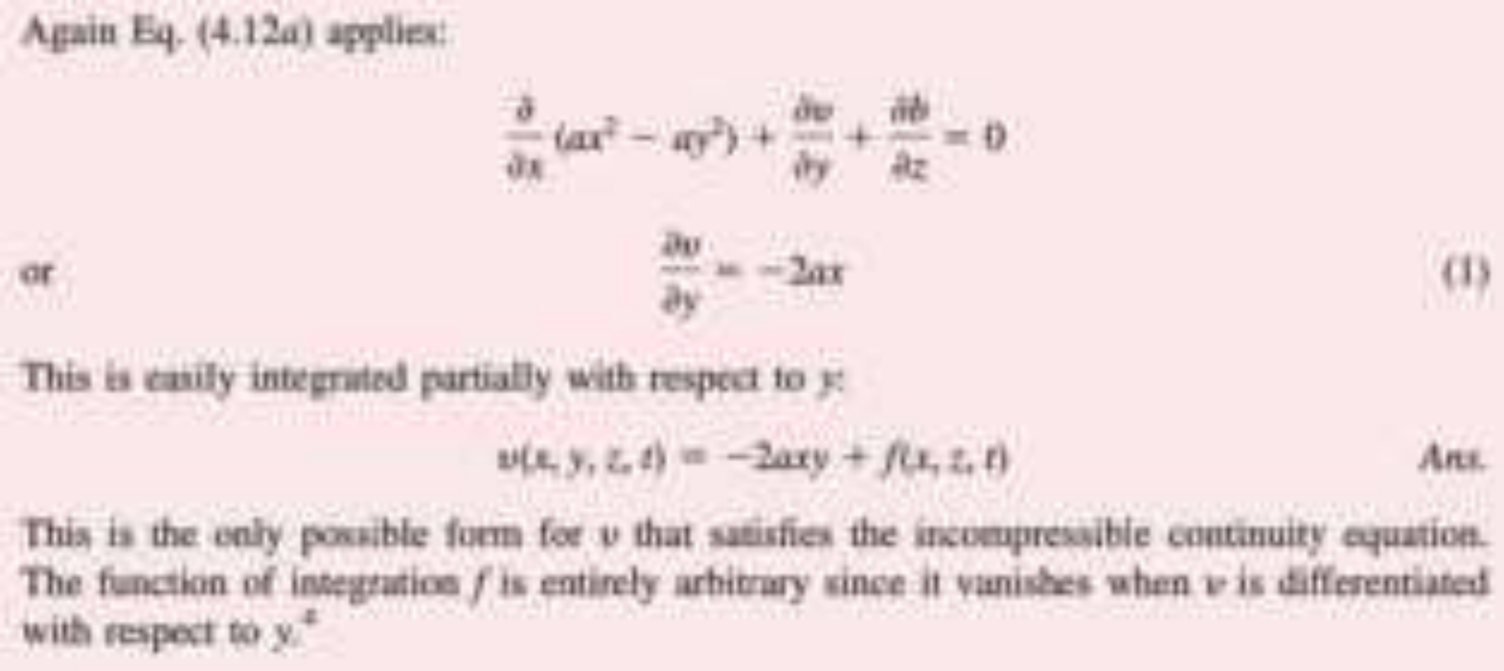
At least two of constants a_1, b_2, and c_3 must have opposite signs. Continuity imposes no restrictions whatever on constants b_1, c_1, a_2, c_2, a_3, and b_3, which do not contribute to a volume increase or decrease of a differential element.



An incompressible velocity field is given by

$$u = a(x^2 - y^2) \qquad v \text{ unknown} \qquad w = b$$

where a and b are constants. What must the form of the velocity component v be?



Again Eq. (4.12*a*) applies:

$$\frac{\partial}{\partial x}(ax^2 - ay^2) + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or

$$\frac{\partial v}{\partial y} = -2ax \qquad (1)$$

This is easily integrated partially with respect to *y*:

$$v(x, y, z, t) = -2axy + f(x, z, t) \qquad \textit{Ans.}$$

This is the only possible form for *v* that satisfies the incompressible continuity equation. The function of integration *f* is entirely arbitrary since it vanishes when *v* is differentiated with respect to *y*.[4]

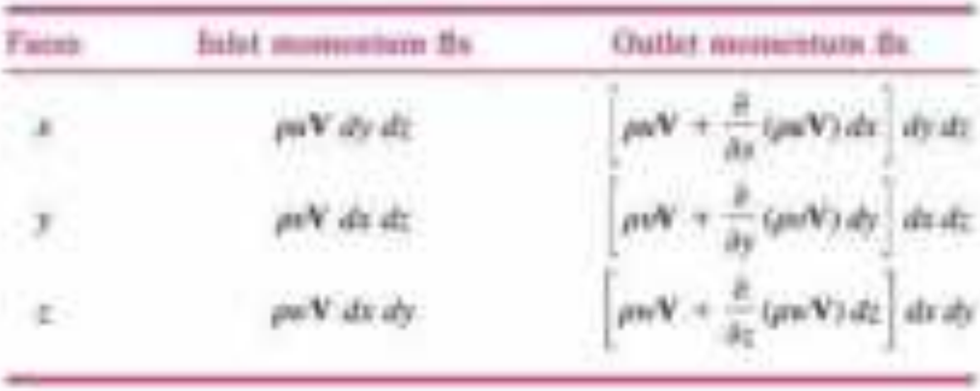
4.3) The Differential Equation for Linear Momentum:

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_{CV} \vec{V}\rho \, d\vartheta + \sum (\dot{m}_i \vec{V}_i)_{out} - \sum (\dot{m}_i \vec{V}_i)_{in}$$

We use the elemental control volume:

$$\frac{\partial}{\partial t}(\vec{V}\rho \, d\vartheta) \approx \frac{\partial}{\partial t}(\rho \vec{V})\, dx\, dy\, dz$$

momentum fluxes:



Faces	Inlet momentum flux	Outlet momentum flux
x	$\rho u \mathbf{V}\, dy\, dz$	$\left[\rho u \mathbf{V} + \frac{\partial}{\partial x}(\rho u \mathbf{V})\, dx\right] dy\, dz$
y	$\rho v \mathbf{V}\, dx\, dz$	$\left[\rho v \mathbf{V} + \frac{\partial}{\partial y}(\rho v \mathbf{V})\, dy\right] dx\, dz$
z	$\rho w \mathbf{V}\, dx\, dy$	$\left[\rho w \mathbf{V} + \frac{\partial}{\partial z}(\rho w \mathbf{V})\, dz\right] dx\, dy$

$$\sum\vec{F} = dx\,dy\,dz\left[\frac{\partial}{\partial t}(\rho\vec{V}) + \frac{\partial}{\partial x}(\rho u\vec{V}) + \frac{\partial}{\partial y}(\rho v\vec{V}) + \frac{\partial}{\partial z}(\rho w\vec{V})\right]$$

$$\frac{\partial}{\partial t}(\rho\vec{V}) + \frac{\partial}{\partial x}(\rho u\vec{V}) + \frac{\partial}{\partial y}(\rho v\vec{V}) + \frac{\partial}{\partial z}(\rho w\vec{V})$$

$$= \vec{V}\left[\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$$

$$+ \rho\frac{\partial\vec{V}}{\partial t} + \rho u\frac{\partial\vec{V}}{\partial x} + \rho v\frac{\partial\vec{V}}{\partial y} + \rho w\frac{\partial\vec{V}}{\partial z}$$

$$= \vec{V}\left[\frac{\partial\rho}{\partial t} + \vec{V}\cdot(\rho\vec{V})\right] + \rho\left[\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}\right]$$

1st term : $\vec{V}$ (Continuity equation) = 0

2nd term : $\rho \frac{d\vec{V}}{dt}$.

Therefore, we have $\Sigma \vec{F} = \rho \frac{d\vec{V}}{dt} dxdydz$

We must consider all the forces acting on the elemental volume.

$d\vec{F} = \vec{f}\, dxdydz$ $\vec{f}$: force per unit volume

As we have seen before, there are two types of forces:

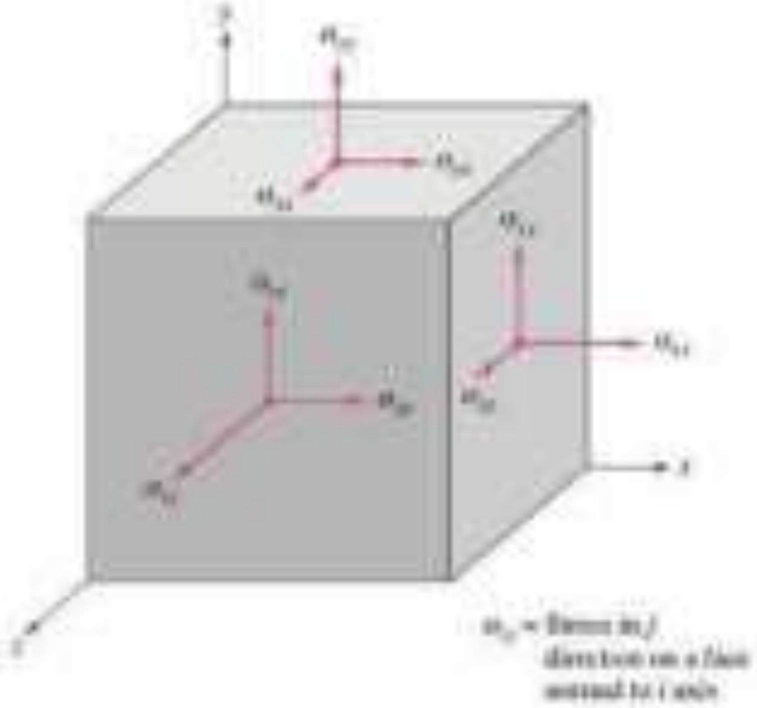
Body forces due to external fields such as gravity, magnetism, electric potential.

$$d\vec{F}_{grav} = \rho \vec{g}\, dx dy dz$$

Most of the time, we take $\vec{g} = -\hat{z} g$ and write

$$d\vec{F}_{grav} = -\rho g\, dx dy dz\, \hat{k}$$

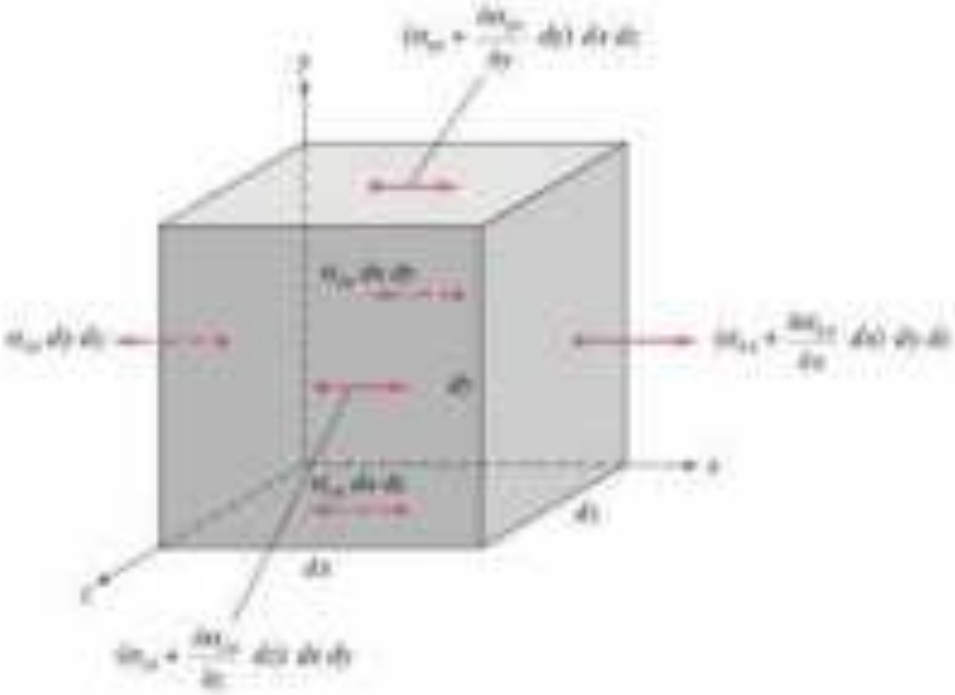
The surface forces are due to stresses on the sides of the control surface.


y
x
z
σ_{ij} = Stress in j direction on a face normal to i axis

They are the sum of the hydrostatic pressure and viscous stresses τ_{ij} that arise from motion with velocity gradients.

$$\sigma_{ij} = \begin{vmatrix} -p + \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & -p + \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & -p + \tau_{zz} \end{vmatrix}$$

Stresses σ_{ij} & τ_{ij} and strain rates ε_{ij} are nine-component tensors.



It is not stresses but their gradients that causes a net force on the differential control surface.

In the figure, we see the stresses contributing in the x-direction.

$$dF_{x,surf} = \left[\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right] dx\,dy\,dz$$

$$= \left[-\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right] dx\,dy\,dz$$

$$\frac{dF_x}{d\vartheta} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad \text{where } d\vartheta = dx\,dy\,dz$$

we can derive tre y & z forces per unit volume similarly.

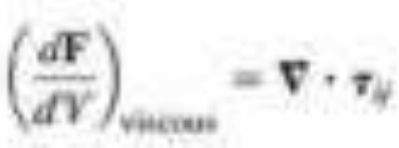
$$\frac{dF_y}{dV} = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}(\tau_{xy}) + \frac{\partial}{\partial y}(\tau_{yy}) + \frac{\partial}{\partial z}(\tau_{zy})$$

$$\frac{dF_z}{dV} = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}(\tau_{xz}) + \frac{\partial}{\partial y}(\tau_{yz}) + \frac{\partial}{\partial z}(\tau_{zz})$$

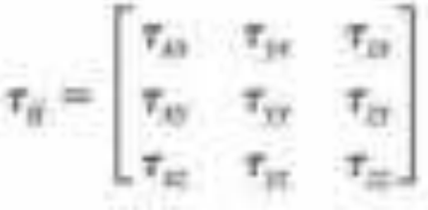
$$\left(\frac{d\mathbf{F}}{dV}\right)_{\text{viscous}} = \mathbf{i}\left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) + \mathbf{j}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) + \mathbf{k}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}\right)$$

$$\left(\frac{d\mathbf{F}}{dV}\right)_{\text{net}} = -\nabla p + \left(\frac{d\mathbf{F}}{dV}\right)_{\text{viscous}}$$

Sometimes we also use the divergence notation:



$$\left(\frac{d\mathbf{F}}{dV}\right)_{\text{viscous}} = \nabla \cdot \tau_{ij}$$

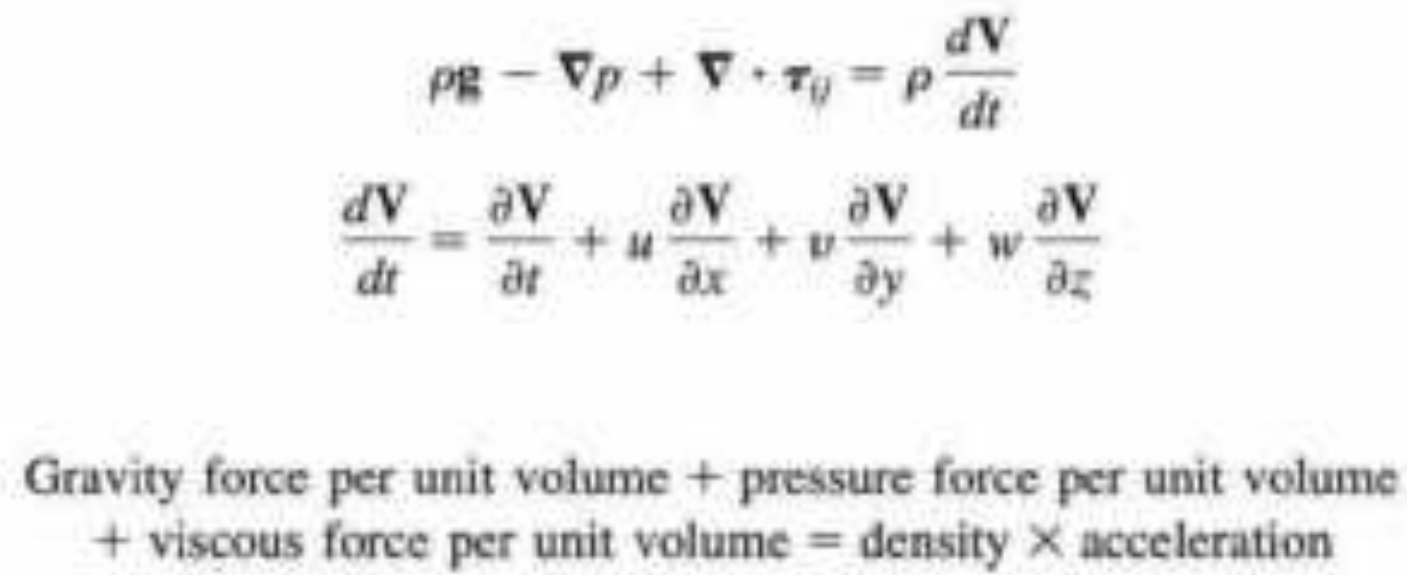


$$\tau_{ij} = \begin{bmatrix} \tau_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \tau_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$$

$$\left(\frac{dF_x}{dV}\right)_{visc} = \vec{\nabla} \cdot \vec{\tau}_{ix} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

$$\left(\frac{dF_y}{dV}\right)_{visc} = \vec{\nabla} \cdot \vec{\tau}_{iy} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial y}$$ etc.

Using our results, we can write the basic differential momentum relation for an inf·simal element:

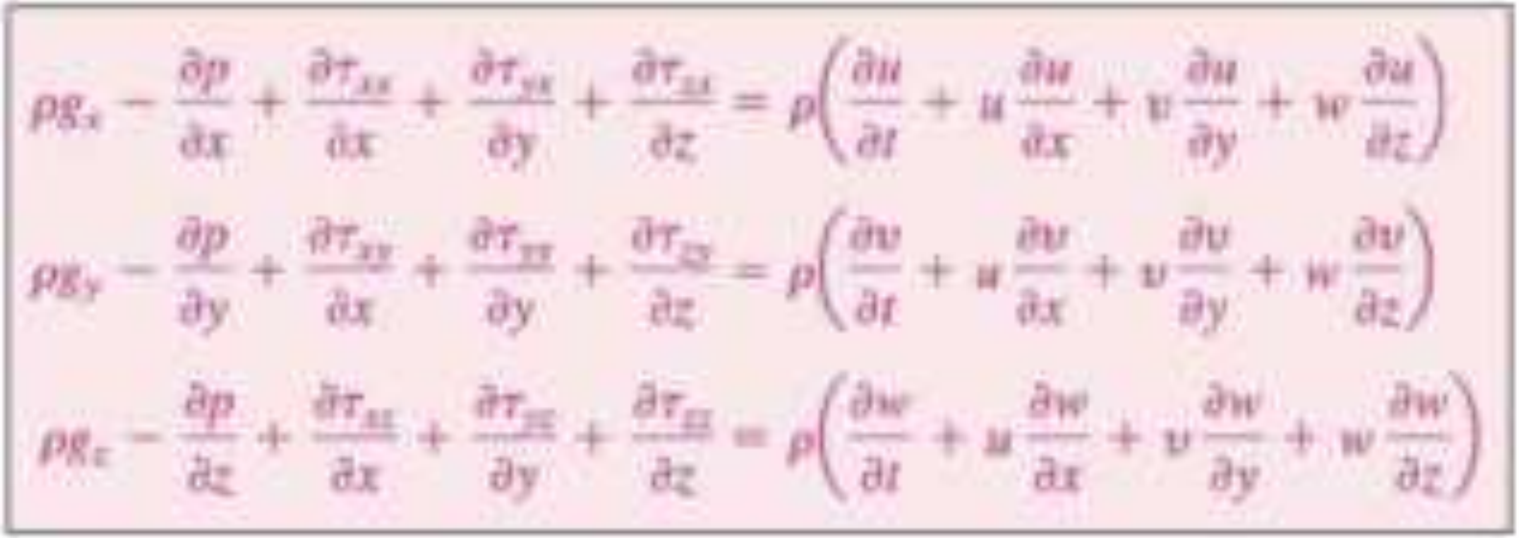


$$\rho\mathbf{g} - \nabla p + \nabla\cdot\tau_{ij} = \rho\frac{d\mathbf{V}}{dt}$$

$$\frac{d\mathbf{V}}{dt} = \frac{\partial\mathbf{V}}{\partial t} + u\frac{\partial\mathbf{V}}{\partial x} + v\frac{\partial\mathbf{V}}{\partial y} + w\frac{\partial\mathbf{V}}{\partial z}$$

Gravity force per unit volume + pressure force per unit volume
+ viscous force per unit volume = density × acceleration

Component equations:



$$\rho g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Note that convective terms produce nonlinearity.

Inviscid Flow: Euler's Equation: No friction: $\vec{\tau}_{ij} = 0$

$$\boxed{\rho \vec{g} - \vec{\nabla} p = \rho \frac{d\vec{V}}{dt}}$$

Euler's eqn. for inviscid flow

We will see that Bernoulli's equation can be obtained by integrating Euler's equation along a stream line.

Newtonian Fluid: Navier-Stokes Equations:

For an incompressible flow, the generalization of $\tau = \mu \frac{du}{dy}$ to 3d viscous flow is

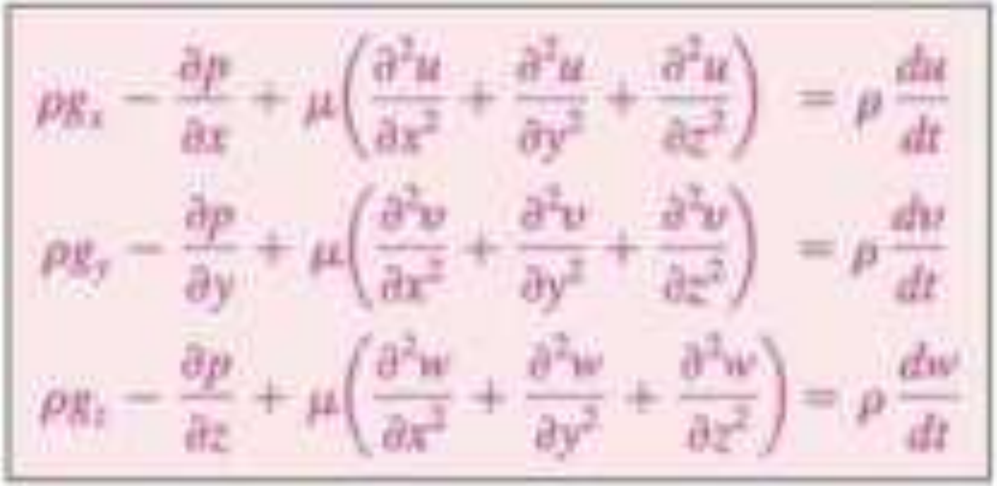
$$\tau_{ij} = \mu \left(\frac{\partial v_i}{\partial r_i} + \frac{\partial v_j}{\partial r_i} \right) \qquad \vec{V} = (u, v, w) \qquad \vec{r} = (x, y, z)$$

$$\tau_{xx} = 2\mu \frac{\partial u}{\partial x} \qquad \tau_{yy} = 2\mu \frac{\partial v}{\partial y} \qquad \tau_{zz} = 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

For a Newtonian fluid with constant density & viscosity:



$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho \frac{du}{dt}$$

$$\rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) = \rho \frac{dv}{dt}$$

$$\rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \rho \frac{dw}{dt}$$

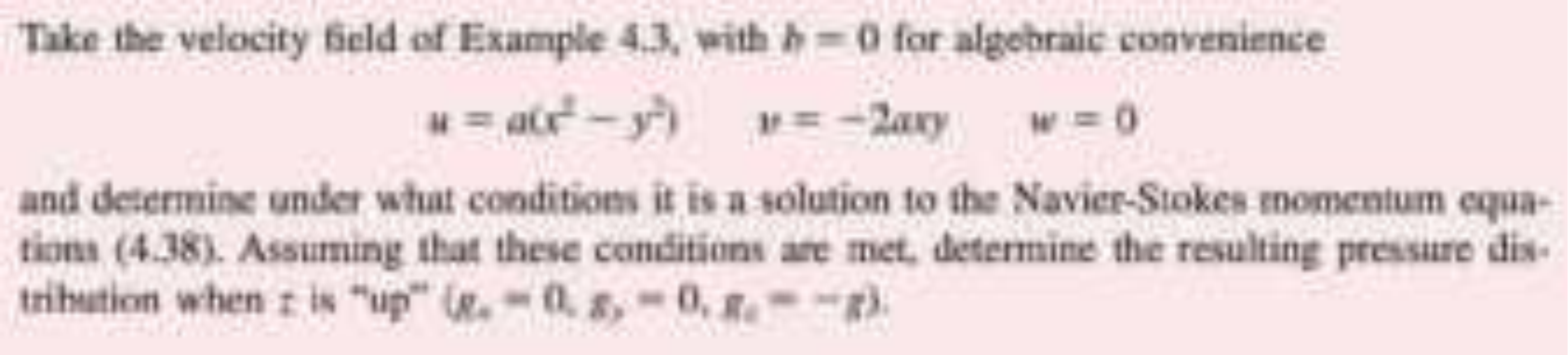
These are the incompressible flow Navier-Stokes equations.

They are 2nd-order non-linear partial differential equations and are quite formiddble. However, solutions have been found to a variety of interesting viscous flow problems.

4 unknowns: p, u, v, w

$$\left.\begin{array}{l}\text{NS eqn.s : 3 eqn.s}\\ \text{Continuity eqn}\end{array}\right\}\ \text{4 equations}$$

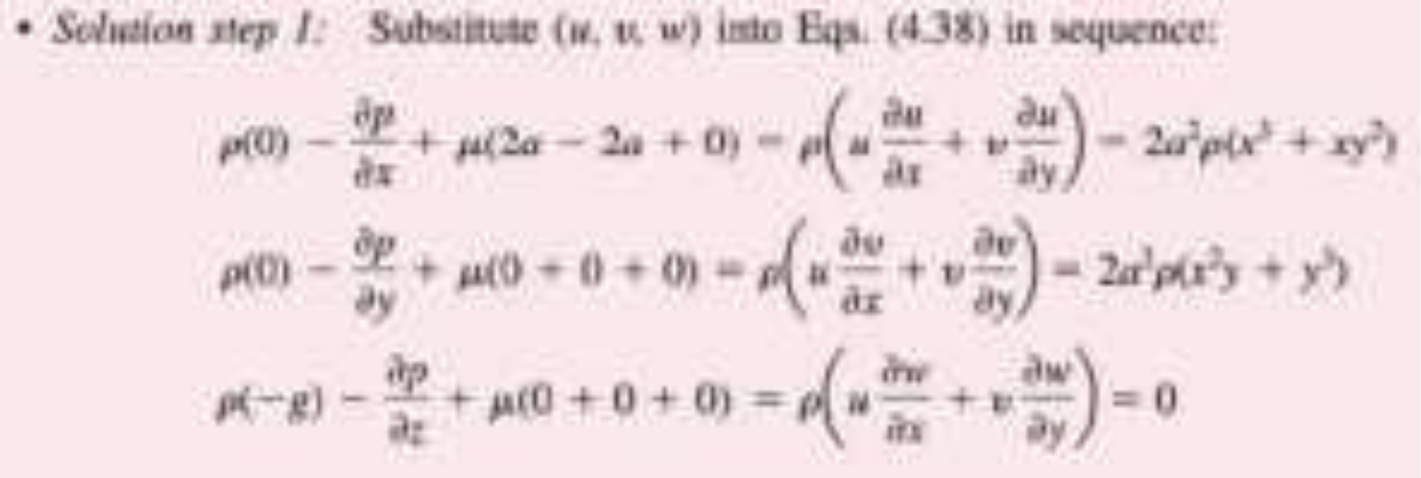
Only a limited number of analytical solutions are known. However, it is possible to obtain approximate but realistic results with computational fluid dynamics.



Take the velocity field of Example 4.3, with $b = 0$ for algebraic convenience

$$u = a(x^2 - y^2) \qquad v = -2axy \qquad w = 0$$

and determine under what conditions it is a solution to the Navier-Stokes momentum equations (4.38). Assuming that these conditions are met, determine the resulting pressure distribution when z is "up" ($g_x = 0$, $g_y = 0$, $g_z = -g$).

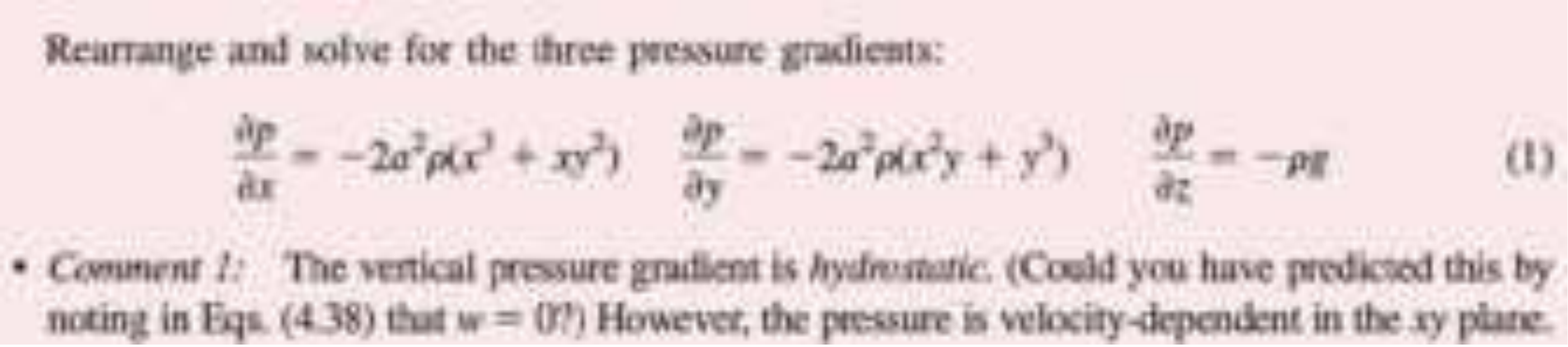


- *Solution step 1:* Substitute (u, v, w) into Eqs. (4.38) in sequence:

$$\rho(0) - \frac{\partial p}{\partial x} + \mu(2a - 2a + 0) = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = 2a^2\rho(x^3 + xy^2)$$

$$\rho(0) - \frac{\partial p}{\partial y} + \mu(0 + 0 + 0) = \rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = 2a^2\rho(x^2y + y^3)$$

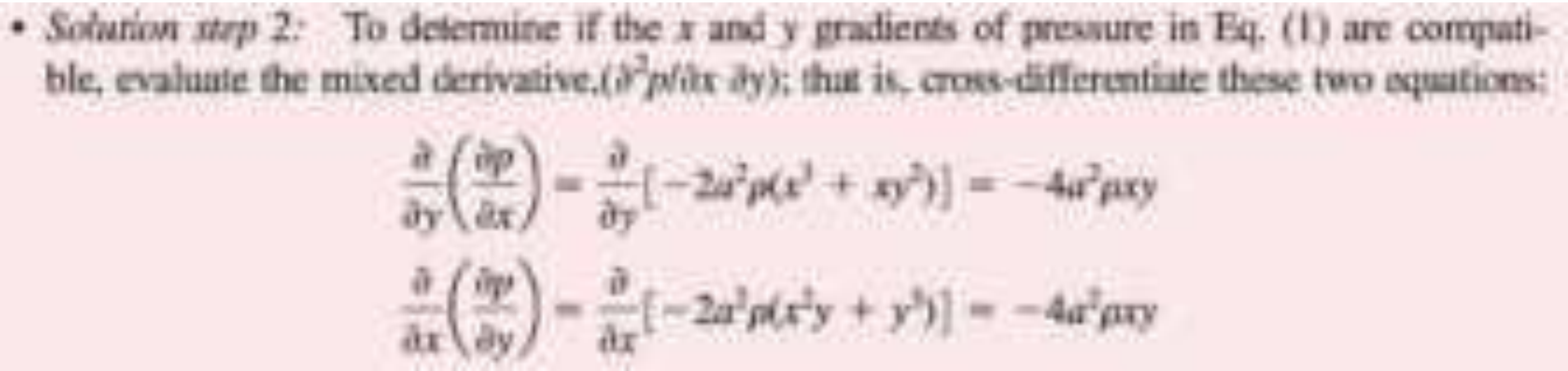
$$\rho(-g) - \frac{\partial p}{\partial z} + \mu(0 + 0 + 0) = \rho\left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y}\right) = 0$$



Rearrange and solve for the three pressure gradients:

$$\frac{\partial p}{\partial x} = -2a^2\rho(x^3 + xy^2) \quad \frac{\partial p}{\partial y} = -2a^2\rho(x^2y + y^3) \quad \frac{\partial p}{\partial z} = -\rho g \qquad (1)$$

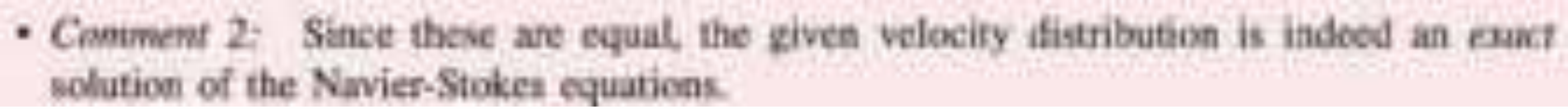
- *Comment 1:* The vertical pressure gradient is *hydrostatic*. (Could you have predicted this by noting in Eqs. (4.38) that $w = 0$?) However, the pressure is velocity-dependent in the xy plane.



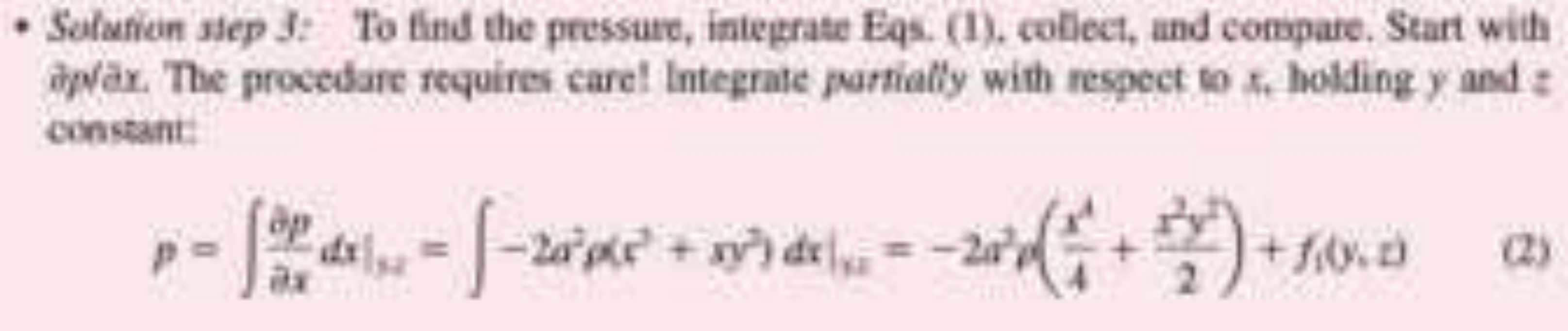
- *Solution step 2:* To determine if the x and y gradients of pressure in Eq. (1) are compatible, evaluate the mixed derivative,($\partial^2 p/\partial x\, \partial y$); that is, cross-differentiate these two equations:

$$\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial y}[-2a^2\rho(x^3 + xy^2)] = -4a^2\rho xy$$

$$\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right) = \frac{\partial}{\partial x}[-2a^2\rho(x^2y + y^3)] = -4a^2\rho xy$$

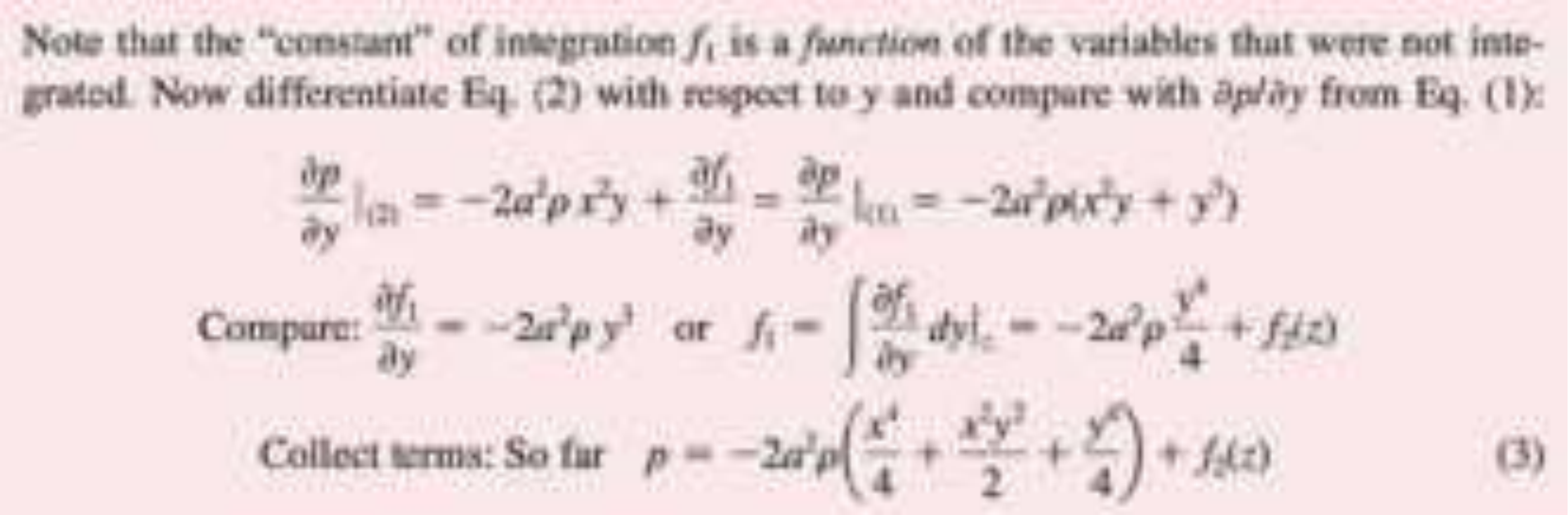


- *Comment 2:* Since these are equal, the given velocity distribution is indeed an *exact* solution of the Navier-Stokes equations.



- *Solution step 3:* To find the pressure, integrate Eqs. (1), collect, and compare. Start with $\partial p/\partial x$. The procedure requires care! Integrate *partially* with respect to x, holding y and z constant:

$$p = \int \frac{\partial p}{\partial x}\,dx\Big|_{y,z} = \int -2a^2\rho(x^3 + xy^2)\,dx\Big|_{y,z} = -2a^2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2}\right) + f_1(y,z) \qquad (2)$$

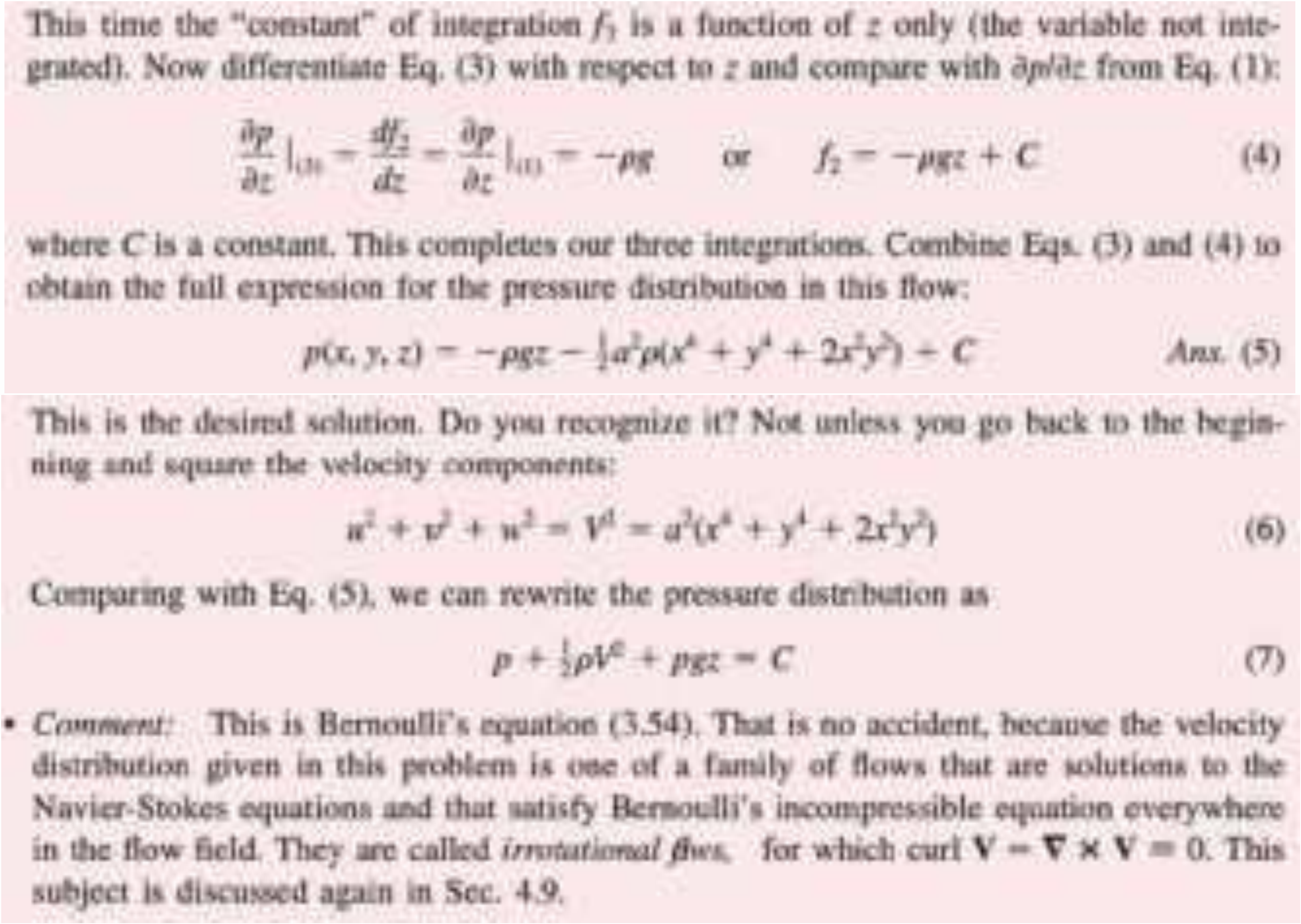


Note that the "constant" of integration f_1 is a *function* of the variables that were not integrated. Now differentiate Eq. (2) with respect to y and compare with $\partial p/\partial y$ from Eq. (1):

$$\frac{\partial p}{\partial y}\Big|_{(2)} = -2a^2\rho\, x^2y + \frac{\partial f_1}{\partial y} = \frac{\partial p}{\partial y}\Big|_{(1)} = -2a^2\rho(x^2y + y^3)$$

$$\text{Compare: } \frac{\partial f_1}{\partial y} = -2a^2\rho\, y^3 \quad \text{or} \quad f_1 = \int \frac{\partial f_1}{\partial y}\,dy\Big|_z = -2a^2\rho\frac{y^4}{4} + f_2(z)$$

$$\text{Collect terms: So far} \quad p = -2a^2\rho\left(\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{y^4}{4}\right) + f_2(z) \qquad (3)$$



This time the "constant" of integration f_2 is a function of z only (the variable not integrated). Now differentiate Eq. (3) with respect to z and compare with $\partial p/\partial z$ from Eq. (1):

$$\frac{\partial p}{\partial z}\Big|_{(3)} = \frac{df_2}{dz} = \frac{\partial p}{\partial z}\Big|_{(1)} = -\rho g \qquad \text{or} \qquad f_2 = -\rho g z + C \tag{4}$$

where C is a constant. This completes our three integrations. Combine Eqs. (3) and (4) to obtain the full expression for the pressure distribution in this flow:

$$p(x, y, z) = -\rho g z - \tfrac{1}{2}a^2\rho(x^4 + y^4 + 2x^2y^2) + C \qquad \text{Ans. (5)}$$

This is the desired solution. Do you recognize it? Not unless you go back to the beginning and square the velocity components:

$$u^2 + v^2 + w^2 = V^2 = a^2(x^4 + y^4 + 2x^2y^2) \tag{6}$$

Comparing with Eq. (5), we can rewrite the pressure distribution as

$$p + \tfrac{1}{2}\rho V^2 + \rho g z = C \tag{7}$$

- *Comment:* This is Bernoulli's equation (3.54). That is no accident, because the velocity distribution given in this problem is one of a family of flows that are solutions to the Navier-Stokes equations and that satisfy Bernoulli's incompressible equation *everywhere* in the flow field. They are called *irrotational flows,* for which curl $\mathbf{V} = \nabla \times \mathbf{V} = 0$. This subject is discussed again in Sec. 4.9.

4.4) The Differential Equation for Angular momentum

We will skip this section.

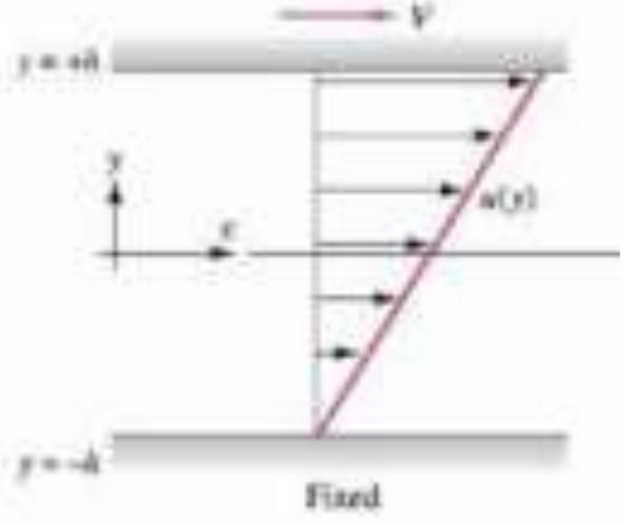
It is enough to know that there is no differential angular momentum equation.

Application of the integral theorem to a differential element yields:

$$\tau_{xz} \approx \tau_{zx}, \quad \tau_{yz} \approx \tau_{zy} \quad \text{etc.}$$

The shear stresses are symmetric: $\tau_{ij} = \tau_{ji}$

Couette Flow between a Fixed & a Moving Plate:



Consider two-dimensional ($\partial/\partial z \to 0$) incompressible viscous flow between parallel plates a distance 2h apart.

When the plates are very wide, the flow is essentially axial $u \neq 0, v = w = 0$

Upper plate moves with velocity V but there is no pressure gradient.

Continuity eqn.: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \frac{\partial u}{\partial x} + 0 + 0 \Rightarrow u = u(y)$ only

Such a flow is said to be fully developed (far downstream of the entrance)

Substitute $u = u(y)$ into the x component of the Navier-Stokes momentum eqn. for 2d flow:

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial x}\right) = -\frac{\partial p}{\partial x} + \rho g_x + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho(0+0) = 0 + 0 + \mu\left(0 + \frac{d^2u}{dy^2}\right) \Rightarrow \frac{d^2u}{dy^2} = 0$$

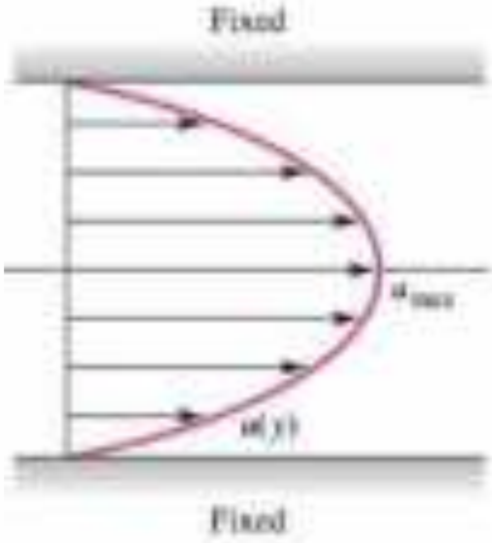
Soln: $u = c_1 y + c_2$

Two constants c_1 & c_2 are found by applying the no-slip condition at the upper and lower plates:

$$\left.\begin{array}{ll} y = +h : & u = V = c_1 h + c_2 \\ y = -h : & u = 0 = c_2(-h) + c_2 \end{array}\right\} \begin{array}{l} C_1 = \dfrac{V}{2h} \\ \\ C_2 = \dfrac{V}{2} \end{array}$$

$$u = \frac{V}{2h} y + \frac{V}{2} \quad -h \leq y \leq h$$: This is the Couette flow.

Flow Due to Pressure Gradient between Two Fixed Plates:



$v = w = 0 \Rightarrow u = u(y)$ from the contunity eqn.

The x-momentum eqn. now gives

$$\rho\left(\cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial x}}\right) = -\frac{\partial p}{\partial x} + \cancel{\rho g_x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\mu\frac{d^2 u}{dy^2} = \frac{\partial p}{\partial z}$$

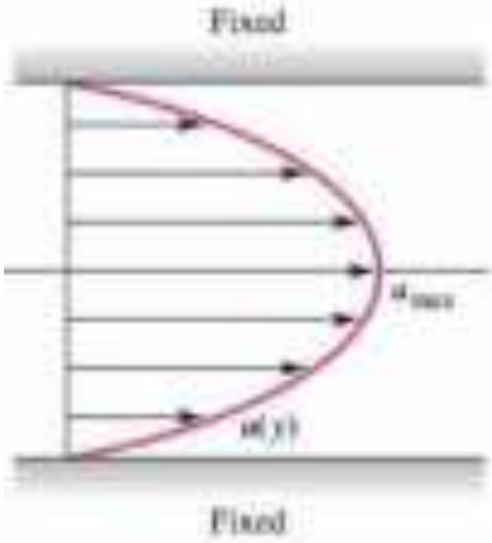
$u = v = 0$. Neglect also gravity: $g_z \approx 0$.

y- & z-momentum eqn.s: $\dfrac{\partial p}{\partial y} = 0 \quad \dfrac{\partial p}{\partial z} = 0 \Rightarrow p = p(x)$

So we have $\mu \dfrac{d^2 u}{dy^2} = \dfrac{dp}{dx}$. LHS = a function of y
RHS = a function of x

Therefore, $\mu \dfrac{d^2 u}{dy^2} = \dfrac{dp}{dx} = \text{constant} < 0$

we can directly conclude that it is a negative constant since the pressure decreases in the flow direction.



u(y) must have a negative curvature everywhere as shown in the figure.

Soln: $u = \frac{1}{\mu}\frac{dp}{dx}y^2 + C_1 y + C_2$

Impose no slip conditions at the walls:

At $y = \pm h$: $u = 0$ $\Rightarrow$ $C_1 = 0$ and $C_2 = -\frac{dp}{dx}\frac{h^2}{2\mu}$

$u = -\frac{dp}{dx}\frac{h^2}{2\mu}\left(1 - \frac{y^2}{h^2}\right)$: Poiseulle parabola

$u_{max} = -\frac{dp}{dx}\frac{h^2}{2\mu}$

Fully Developed Laminar Pipe Flow:

(The most useful exact result of NS eqn's)

Fully developed: The region studied is far enough from the entrance that the flow is purely axial: $V_r = 0 \quad V_\theta = 0 \quad V_z \neq 0$

$$\frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(v_\theta) + \frac{\partial}{\partial z}(v_z) = 0$$

The r-momentum equation:

$$\frac{\partial v_r}{\partial t} + (\mathbf{V}\cdot\nabla)v_r - \frac{1}{r}v_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r + \nu\left(\nabla^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right)$$

The θ-momentum equation:

$$\frac{\partial v_\theta}{\partial t} + (\mathbf{V}\cdot\nabla)v_\theta + \frac{1}{r}v_r v_\theta = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + g_\theta + \nu\left(\nabla^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right)$$

The z-momentum equation:

$$\frac{\partial v_z}{\partial t} + (\mathbf{V}\cdot\nabla)v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z + \nu\nabla^2 v_z$$

For equations of motion in cylindrical coordinates, see Appendix D of your book.

Continuity eqn: $\frac{\partial}{\partial z}(v_z) = 0 \Rightarrow v_z = v_z(r)$

r-momentum eqn.: $\frac{\partial p}{\partial r} = 0 \Rightarrow p = p(z)$

z-momentum eqn.: $\rho v_z \frac{\partial v_z}{\partial z} = -\frac{dp}{dz} + \mu \nabla^2 v_z$

$$= -\frac{dp}{dz} + \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)$$

LHS = 0 due to continuity eqn.

$\frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right) = \frac{dp}{dz}$ = constant < 0 Again, the seperation constant is negative.

$$v_z = \frac{dp}{dz}\frac{r^2}{4\mu} + c_1 \ln r + c_2 .$$

Boundary conditions: i) no slip at the wall.
ii) finite velocity at the centerline.

i) $r = R$, $v_z = 0 = \frac{dp}{dz}\frac{R^2}{4\mu} + c_1 \ln R + c_2$

ii) $r = 0$, $v_z = \text{finite} = 0 + c_1 \ln(0) + c_2 \Rightarrow c_1 = 0$

$$c_2 = -\frac{dp}{dz}\frac{1}{4\mu}(R^2 - r^2)$$

$$v_z = -\frac{dp}{dz}\frac{1}{4\mu}(R^2 - r^2)$$: Fully developed Hagen_Poiseuille flow

The velocity profile is a paraboloid with a maximum at the centerline. Just as in Example 4.10, knowledge of the velocity distribution enables other parameters to be calculated:

$$V_{max} = u_z(r = 0) = \left(-\frac{dp}{dz}\right)\frac{R^2}{4\mu}$$

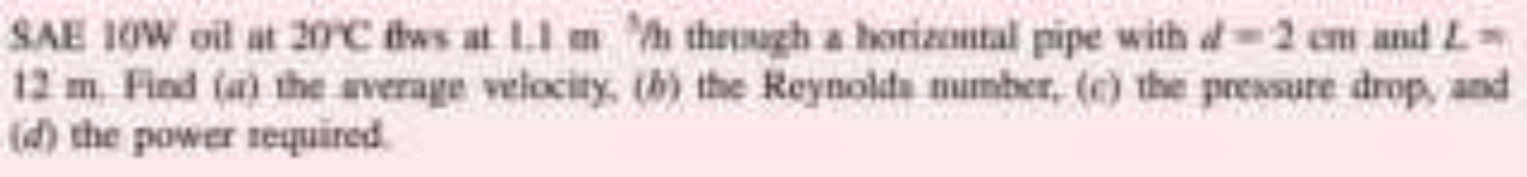
$$V_{avg} = \frac{1}{A}\int u_z\,dA = \frac{1}{\pi R^2}\int_0^R V_{max}\left(1-\frac{r^2}{R^2}\right)2\pi r\,dr = \frac{V_{max}}{2} = \left(-\frac{dp}{dz}\right)\frac{R^2}{8\mu}$$

$$Q = \int u_z\,dA = \int_0^R V_{max}\left(1-\frac{r^2}{R^2}\right)2\pi r\,dr = \pi R^2 V_{avg} = \frac{\pi R^4}{8\mu}\left(-\frac{dp}{dz}\right) = \frac{\pi R^4\,\Delta p}{8\mu\,L}$$

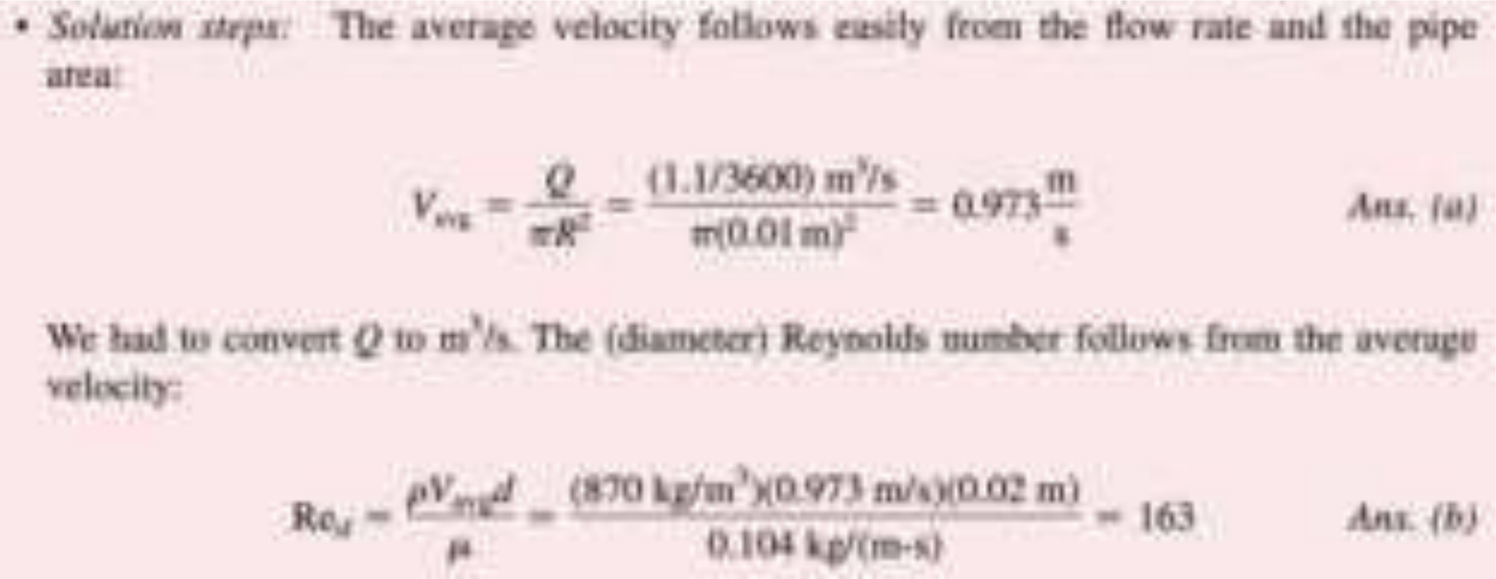
$$\tau_{wall} = \mu\left|\frac{\partial u_z}{\partial r}\right|_{r=R} = \frac{4\mu V_{avg}}{R} = \frac{R}{2}\left(-\frac{dp}{dz}\right) = \frac{R}{2}\frac{\Delta p}{L} \qquad (4.138)$$

Note that we have substituted the equality $(-dp/dz) = \Delta p/L$, where Δp is the pressure drop along the entire length L of the pipe.

These formulas are valid as long as the flow is *laminar*—that is, when the dimensionless Reynolds number of the flow, $\mathrm{Re}_D = \rho V_{avg}(2R)/\mu$, is less than about 2100. Note also that the formulas do not depend on density, the reason being that the convective acceleration of this flow is zero.



SAE 10W oil at 20°C flows at 1.1 m^3/h through a horizontal pipe with $d = 2$ cm and $L =$ 12 m. Find (*a*) the average velocity, (*b*) the Reynolds number, (*c*) the pressure drop, and (*d*) the power required.

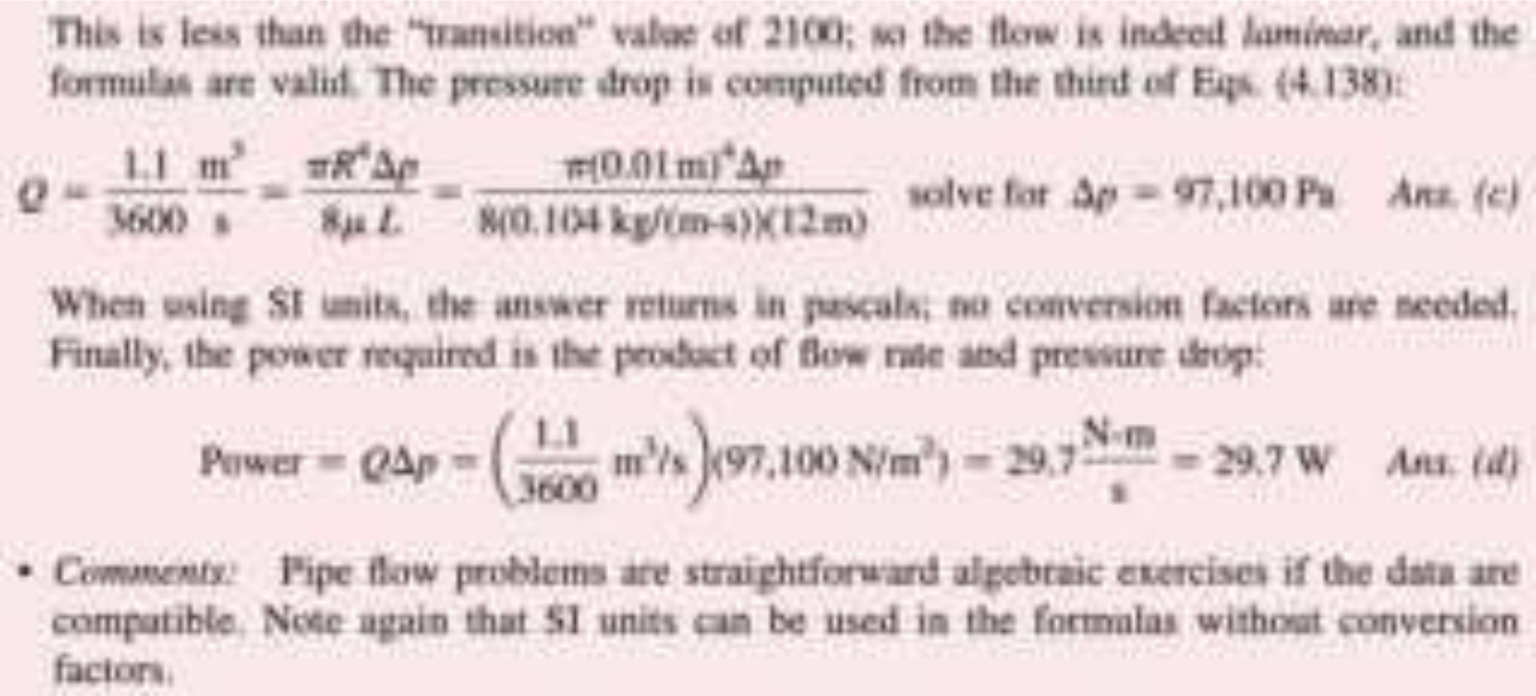


- *Solution steps:* The average velocity follows easily from the flow rate and the pipe area:

$$V_{avg} = \frac{Q}{\pi R^2} = \frac{(1.1/3600)\ \text{m}^3/\text{s}}{\pi (0.01\ \text{m})^2} = 0.973\,\frac{\text{m}}{\text{s}} \qquad \textit{Ans. (a)}$$

We had to convert Q to m^3/s. The (diameter) Reynolds number follows from the average velocity:

$$\text{Re}_d = \frac{\rho V_{avg} d}{\mu} = \frac{(870\ \text{kg/m}^3)(0.973\ \text{m/s})(0.02\ \text{m})}{0.104\ \text{kg/(m-s)}} = 163 \qquad \textit{Ans. (b)}$$



This is less than the "transition" value of 2100; so the flow is indeed *laminar*, and the formulas are valid. The pressure drop is computed from the third of Eqs. (4.138):

$$Q = \frac{1.1}{3600}\frac{\text{m}^3}{\text{s}} = \frac{\pi R^4 \Delta p}{8\mu L} = \frac{\pi(0.01\ \text{m})^4 \Delta p}{8(0.104\ \text{kg/(m-s)})(12\ \text{m})} \quad \text{solve for } \Delta p = 97{,}100\ \text{Pa} \qquad \textit{Ans. (c)}$$

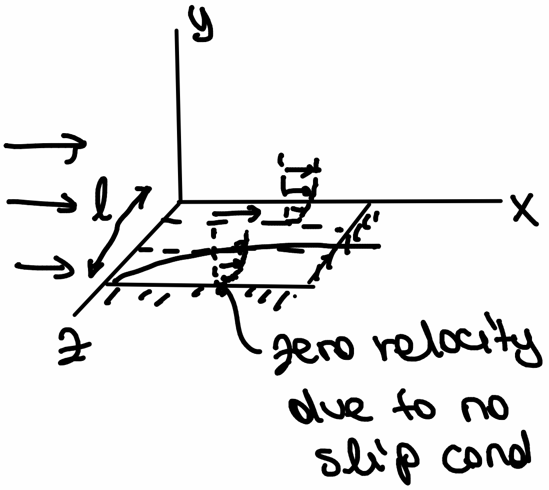
When using SI units, the answer returns in pascals; no conversion factors are needed. Finally, the power required is the product of flow rate and pressure drop:

$$\text{Power} = Q\Delta p = \left(\frac{1.1}{3600}\ \text{m}^3/\text{s}\right)(97{,}100\ \text{N/m}^2) = 29.7\frac{\text{N-m}}{\text{s}} = 29.7\ \text{W} \qquad \textit{Ans. (d)}$$

- *Comments:* Pipe flow problems are straightforward algebraic exercises if the data are compatible. Note again that SI units can be used in the formulas without conversion factors.

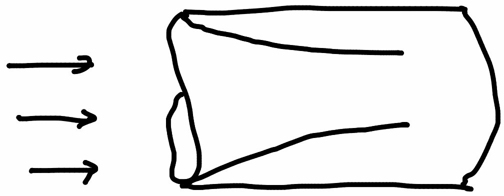
Concept of a Boundary Layer

Boundary layers could form at different places. But, we will consider a flat plate first for its simplicity.

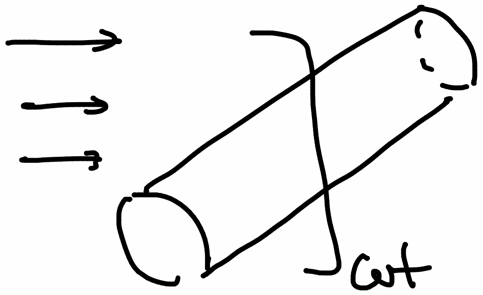


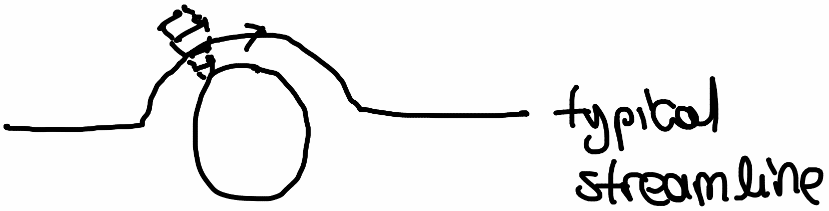
Every time we draw a dist., you should always remember that we are looking at a particular section.

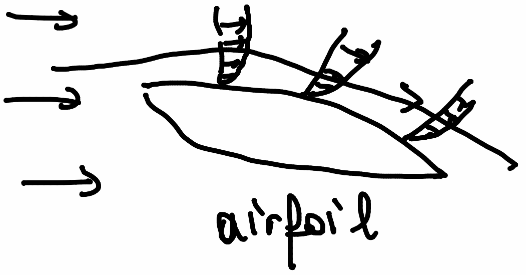
What you see in the books is a 2d boundary layer. It is assumed that nothing changes as we move along the z-direction. This essentially means all the fluid properties is a function of x&y only. Ex: $\vec{V} = \vec{V}(x,y)$, $P = P(x,y)$ $\rho = \rho(x,y)$
This becomes true when we have a very large plate $(l >> 1)$.



A boundary layer (BL) is developed in a viscous flow through a pipe too.

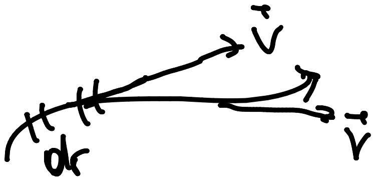






We have a similar velocity profile for all geometries

Reminder: Streamline: a line to which the velocity vector is always tangent



$$\vec{V} = u\hat{i} + v\hat{j} + w\hat{k}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$
$$\vec{V} \parallel d\vec{r}$$

Eqn. for a streamline:
$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \frac{dr}{V} \qquad V = \sqrt{u^2+v^2+w^2}$$

2d flows: $\left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u}$: This diff. eqn. can be solved in many cases.

Ex: $\vec{V} = \left[(1+2x)\hat{i} + (2+y)\hat{j}\right]$ m/s

Find the stagnation point, i.e., the point at which $\vec{V} = 0$, and the eqn. describing streamlines.

Hint: $\int dt \frac{1}{at+b} = \frac{1}{a}\ln(|at+b|) + C$

Solution: stagnation point $\vec{V} = 0$

$u = 1+2x$
$v = 2+y$

$u = 1+2x \Rightarrow x = -0.5$
$v = 2+y \Rightarrow y = -2$

The stagnation point is $(-0.5, 2)$.

$$\left(\frac{dy}{dx}\right)_{\text{along a streamline}} = \frac{v}{u}$$

$$\frac{dy}{dx} = \frac{2+y}{1+2x}$$; seperable eqn, i.e., $\int \frac{dx}{f(x)} = \int \frac{dy}{g(y)}$

$$\int \frac{dy}{y+2} = \int \frac{dx}{2x+1} \Rightarrow \ln(y+2) = \frac{1}{2}\ln(2x+1) + \ln C$$

$$= \ln\left[C(2x+1)^{1/2}\right]$$

$$y = C(2x+1)^{1/2} - 2$$ C: integration constant

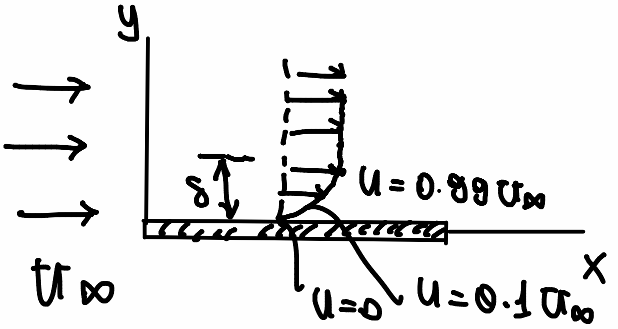
$$(y+2)^2 = C^2(2x+1)$$ Each value of C generates a different streamline

→ End of Review.

The reason why we have boundary layers is best explained by Prandtl's boundary layer theory. It is developed due to viscous effects.

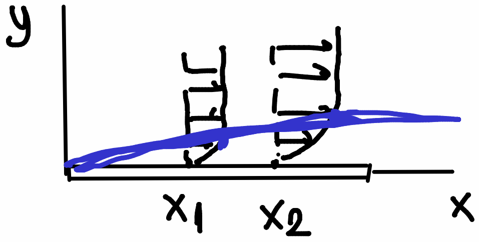
viscosity: a property of the fluid

⇒ no slip cond ⇒ non-trivial velocity profile near solid boundaries.



As we go away from the plate, fluid layers are moving more freely.

Beyond the distance δ, the viscous effects vanish and we recover the free stream properties. This is the boundary layer.



$u(x_1,\delta_1) = u(x_2,\delta_2) \approx U_\infty$

As we move along the x-direction, we need a larger δ to recover the free stream speed $\underline{(\delta_2 > \delta_1)}$

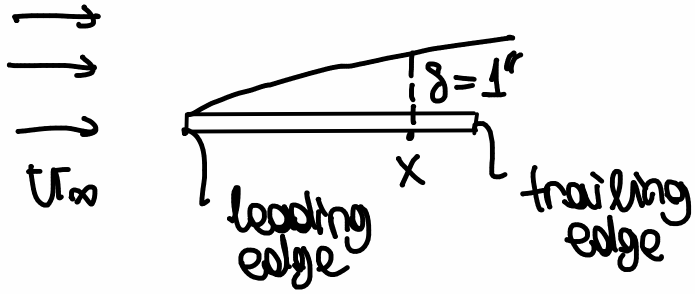
Outside the boundary layer, the viscous effects are not present. (That does not mean $\mu=0$) Therefore, it's a potential flow.

Vorticity: $\vec{\omega} = \vec{\nabla} \times \vec{V}$ inviscid flow: $\vec{\omega} = 0$

Then, we can define a potential function ϕ as
$\vec{V} = \vec{\nabla}\phi$ ϕ: velocity potential HW: $\vec{\nabla} \times \vec{\nabla}\phi = 0$

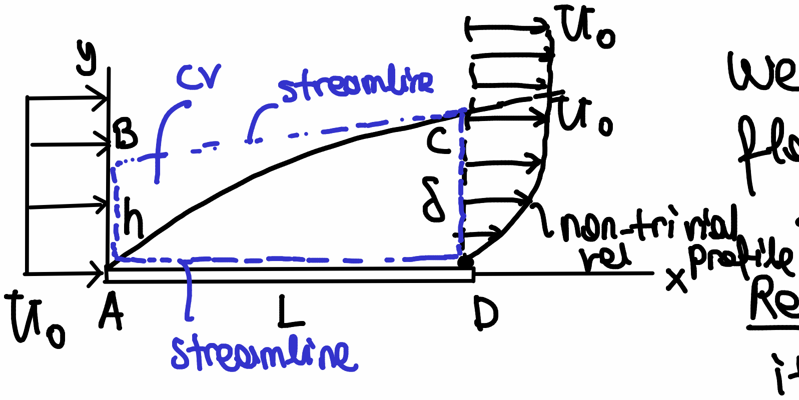
Solving diff. eqn's for ϕ only instead of the velocity comp.s (u,v,w) is much easier, especially in 2d.

Problem: A long thin flat plate is placed parallel to a 6m/s free stream of water at 20°C. At what distance x from the leading edge, will the BL thickness be 1inch.
Given at 20°C, $\nu_{water} = 1.01 \times 10^{-6}\ m^2/s$



There are different terminologies to characterise a BL: BL thickness, displacement thickness, momentum thickness, ...

We are going to do a control volume analysis.



We cannot have a flow across a streamline

Remember: It's a plate, it has a width.

AB: A(0,0) B(0,h) $\vec{V}\cdot\vec{n} = -U_0$

BC: B(0,h) C(L,δ) $\vec{V}\cdot\vec{n} = 0$ since BC is a streamline, i.e., no flow across it.

CD: C(L,δ) D(L,0) $\vec{V}\cdot\vec{n} = u(y)$ + sign $\vec{V}$ & $\vec{n}$ are in the same dir.

AD: A(0,0) D(L,0) $\vec{V}\cdot\vec{n} = 0$ since AD is a streamline

The force acting on the cv is the drag force

Total pressure force is zero. We have learned why.

$$\Sigma F_x = -D = \int_{AB} \rho u (\vec{V}\cdot\vec{n})dA + \int_{CD} \rho u (\vec{V}\cdot\vec{n})dA$$

$dA = b\,dy$ where b is the width. $\rho = \text{const.}$

$$-D = \rho b \int_0^h U_0(-U_0)dy + \rho b \int_0^\delta u\,u\,dy$$

$$= -\rho U_0^2 b h + \rho b \int_0^\delta u^2(y)dy$$

$$\boxed{D = \rho U_0^2 b h - \rho b \int_0^\delta u^2(y)\,dy}$$ ①

we obtained an exp. for the drag force.

Cons. of mass : $\rho \int_{CS} (\vec{V}\cdot\vec{n})\,dA = 0$

$$\int_{AB} (\vec{V}\cdot\vec{n})\,dA + \int_{CD} (\vec{V}\cdot\vec{n})\,dA = 0$$

$$b\int_0^h (-U_0)\,dy + b\int_0^\delta u(y)\,dy = 0$$ b's cancel out

$$\boxed{h U_0 = \int_0^\delta u(y)\,dy}$$ ②

Use ② in ①

$$D = b\rho U_0 \int_0^\delta u(y)\,dy - b\rho \int_0^\delta u^2(y)\,dy$$

$$\boxed{D = b\rho \int_0^\delta u(y)\left[U_0 - u(y)\right] dy}$$ ③

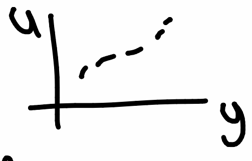
First proposed by Theodore von Karman in 1921.

It's a momentum integral theory for BLs.

If we have some idea of u(y), we can estimate the drag force by using eqn. (3).

But, what is the velocity profile?
There are some proposed approximations.

Make an exp. → data po.
Use diff. analysis → numerical soln.



Write down a function u(y), which is a good app.

Let's use $u = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)$.

Let $\eta = \frac{y}{\delta}$. Then, $dy = \delta\, d\eta$ $\quad y: 0 \to \delta$

$\eta: 0 \to 1$

$u(\eta) = U_0(2\eta - \eta^2)$

$$\boxed{D = b\rho\delta \int_0^1 u(\eta)\left[U_0 - u(\eta)\right] d\eta}$$

Drag force for any $u(\eta)$ where $\eta = \frac{y}{\delta}$

$$D = b\rho\delta \int_0^1 U_0(2\eta - \eta^2)\left[U_0 - U_0(2\eta - \eta^2)\right] d\eta$$

$$= b\rho\delta U_0^2 \int_0^1 (2\eta - \eta^2)(1 - 2\eta + \eta^2)\, d\eta$$

$$= b\rho\delta U_0^2 \int_0^1 (2\eta - 4\eta^2 + 2\eta^3 - \eta^2 + 2\eta^3 - \eta^4)\, d\eta$$

$$= b\rho\delta U_0^2 \int_0^1 (2\eta - 5\eta^2 + 4\eta^3 - \eta^4)\, d\eta$$

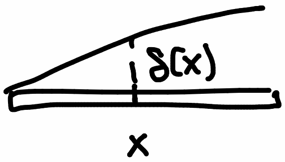
$$= b\rho\delta U_0^2 \left(\eta^2 - \frac{5}{3}\eta^3 + \eta^4 - \frac{1}{5}\eta^5\right)\Big|_0^1$$

$$= b\rho\delta U_0^2 \left(1 - \frac{5}{3} + 1 - \frac{1}{5}\right)$$

$$\boxed{D = \frac{2}{15}\rho b \delta U_0^2 \quad \text{for} \quad u = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right)} \quad ④$$

At each value of x, we have a different $\delta = \delta(x)$

Therefore, $\boxed{D(x) = \frac{2}{15}\rho b U_0^2 \delta(x)}$ ~ ⑤



How do we relate it to the shear stress on the wall?

$$\boxed{D = \int \tau_w \, dA \;=\; b\int_0^x \tau_w(x)\,dx}$$ ~ ⑥

From ⑥ : $\tau_w = \frac{1}{b}\frac{dD}{dx}$. Use the drag force that we have obtained.

$$\tau_w = \frac{1}{\cancel{b}}\frac{2}{15}\rho \cancel{b} U_0^2 \frac{d}{dx}\delta(x) \rightarrow \boxed{\tau_w = \frac{2}{15}\rho U_0^2 \frac{d}{dx}\delta(x)}$$ ~ ⑦

But, we also know from Newton's law of viscosity

$$\tau_w = \mu \frac{du}{dy}\Big|_{y=0} \qquad u = U_0\left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) \qquad \frac{du}{dy} = U_0\left(\frac{2}{\delta} - \frac{2y}{\delta^2}\right)$$

$$\frac{du}{dy}\Big|_{y=0} = \frac{2U_0}{\delta}$$

Therefore, we obtain $\boxed{\tau_w = \frac{2\mu U_0}{\delta}}$ ~ ⑧

Insert ⑧ into ⑦

$$\frac{2\mu U_0}{\delta} = \frac{2}{15}\rho U_0^2 \frac{d\delta}{\delta x} \Rightarrow \frac{15\nu}{U_0}dx = \delta d\delta$$

↓

seperable eqn.

$$\Rightarrow \frac{15\nu}{U_0}x = \frac{1}{2}\delta^2$$

No integration constant is needed since $\delta = 0$ when $x = 0$.

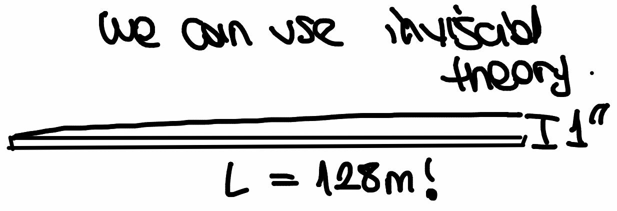
$$\frac{\delta^2}{x^2} = \frac{30\nu}{U_0 x} = \frac{30}{Re_x}$$

$$\boxed{\frac{\delta}{x} = \sqrt{\frac{30}{Re_x}} \approx \frac{5.5}{\sqrt{Re_x}}} \sim ⑨$$

HW: If you use this eqn with $\delta = 1''$, you find $x = 127.75$ m!

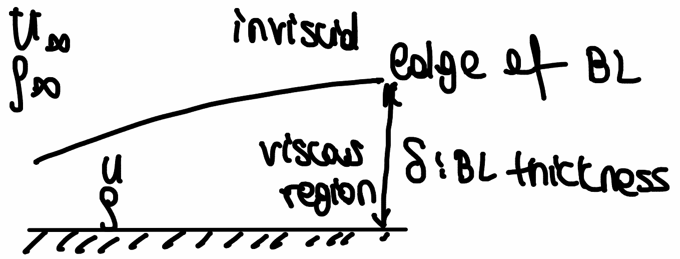
> Caution: You should be able to derive these relations for a different approx. for $u(y)$.

To have a BL thickness of 1", you need to travel a distance of nearly 128 m! This means the region where viscous effects dominate is quite small.



However, to calculate the drag force, this analysis is a must!

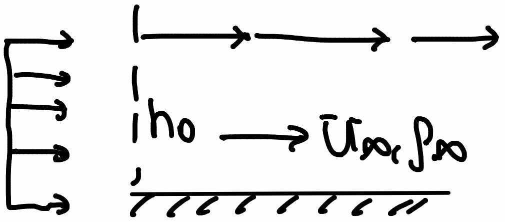
Boundary layer displacement thickness



Fluid is viscous all the time. But, outside the BL we can take $\nu = 0$

Let us think of two particular cases:

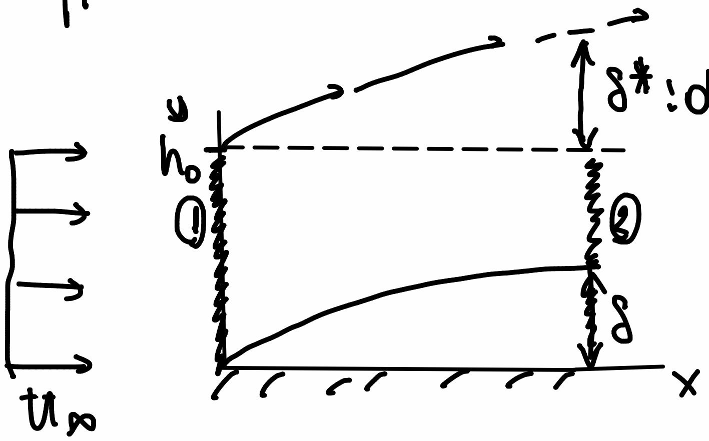
1) Hypothetical case: No BL we neglect the viscosity effects.



How a streamline would look like at the height h_0

inviscid case

2) How does this picture change when viscous effects are taken into account?



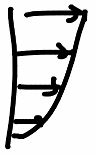
Streamlines will be shifted away from the horizontal line

δ^*: boundary layer displacement thickness, displacement from the horizontal line.

$\delta^* = \delta$ or not? We need to calculate & see.

A = actual mass flow through the crossection ② per unit depth

$$= \int_0^{h_0} \rho u \, dy$$



$u(y)$: velocity profile at the exit.

B = hypothetical mass flow across ② per unit depth

$$= \int_0^{h_0} \rho_\infty U_\infty \, dy = \rho_\infty U_\infty h_0$$

Clearly, $B > A$ since $U_\infty > u(y)$ inside the BL

$B - A$ = missing mass flow due to BL

$$= \int_0^{h_0} \rho_\infty U_\infty \, dy - \int_0^{h_0} \rho u \, dy = \int_0^{h_0} (\rho_\infty U_\infty - \rho u) \, dy$$

We can write down the missing mass flow in terms δ^* : $B - A = \rho_\infty U_\infty \delta^*$

$$\rho_\infty U_\infty \delta^* = \int_0^{h_0} (\rho_\infty U_\infty - \rho u) \, dy$$

$$\boxed{\delta^* = \int_0^{h_0} \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right) dy} \sim ①$$

If h_0 is just outside the BL, i.e., $h_0 \approx \delta$ and we use the velocity profile

$$u(y) = U_\infty \left(\frac{2y}{\delta} - \frac{y^2}{\delta^2}\right) = U_\infty (2\eta - \eta^2) \text{ with } \eta = \frac{y}{\delta}$$

$$\frac{u}{U_\infty} = 2\eta - \eta^2$$

$dy = \delta \, d\eta$ $\quad y : 0 \to \delta \quad \eta : 0 \to 1$

Incompressible flow, $\rho = \rho_\infty$.

Use ① now:

$$\delta^* = \int_0^{h_0} \left(1 - \frac{\rho u}{\rho_\infty U_\infty}\right) dy = \delta \int_0^1 \left(1 - \frac{u}{U_\infty}\right) d\eta$$

$$= \delta \int_0^1 \left[1 - (2\eta - \eta^2)\right] d\eta = \delta \int_0^1 (1 - 2\eta + \eta^2)\, d\eta$$

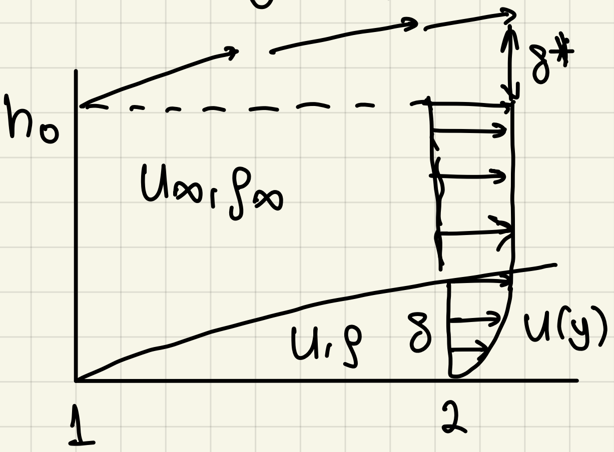
$$= \delta \left(\eta - \eta^2 + \frac{1}{3}\eta^3\right) = \delta \left(1 - 1 + \frac{1}{3}\right)$$

$$\boxed{\delta^* = \frac{1}{3}\delta} \text{ ②} \qquad \text{for } u(\eta) = U_\infty(2\eta - \eta^2)$$

Remember $\quad \dfrac{\delta}{x} = \sqrt{\dfrac{30}{Re_x}} \approx \dfrac{5.5}{\sqrt{Re_x}}$

$$\boxed{\frac{\delta^*}{x} = \sqrt{\frac{10}{3\,Re_x}} \approx \frac{1.83}{\sqrt{Re_x}}}$$

Boundary Layer Momentum Thickness (θ):



$u(y) =$

Whatever the mass enters at section 1, it needs to leave at section 2. It needs more height to compensate for $u(y) < U_\infty$.

Whenever a BL exists, it displaces or pushes your streamlines away from itself. It acts almost like a solid & streamlines gets displaced. To allow the incoming mass to exit, we need the additional height δ^*. Clearly, there is also a change in momentum at the height h_0. Physically, this means we need a height difference to allow the incoming mom. to exit.

$$\dot{m} = \int \rho (\vec{v} \cdot \vec{n}) dA \qquad \text{momentum flux} = \int \rho \vec{v} (\vec{v} \cdot \vec{n}) dA$$

A = momentum flux per unit depth

$= \rho u^2 dy$

$dA = b\,dy$
$b = 1$

B = hypothetical momentum flux

$= U_\infty (\rho u dy) = \rho U_\infty u dy$

$$B - A = \int_0^{h_0} \rho u (U_\infty - u) dy \equiv U_\infty (\rho_\infty U_\infty \theta) = \rho_\infty U_\infty^2 \theta$$

$$\boxed{\theta = \int \frac{\rho}{\rho_\infty}\frac{u}{U_\infty}\left(1-\frac{u}{U_\infty}\right)dy}$$

Again, if we know $u(y)$, we can find θ.

Let us calculate θ for $u = U_\infty(2\eta-\eta^2)$ with $\eta = \frac{y}{\delta}$

Taking $h_0 = \delta$, we have $dy = \delta\, d\eta$ $\quad y: 0 \to \delta$

$\eta: 0 \to 1$

$$\theta = \delta \int_0^1 (2\eta-\eta^2)(1-2\eta+\eta^2)\,d\eta = \frac{2}{15}\delta$$

For this velocity dist, we found $\frac{\delta}{x} = \sqrt{\frac{30}{Re_x}}$.

Therefore, $\frac{\theta}{x} = \frac{2}{15}\sqrt{\frac{30}{Re_x}} \approx \frac{0.730}{\sqrt{Re_x}}$

Let's also calculate the drag coef. C_D and the skin friction coef. C_F for this velocity dist.

We have found

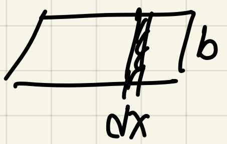
$$D = \frac{2}{15}\rho b \delta U_\infty^2 \qquad \tau_w = \frac{2\mu U_\infty}{\delta}$$

$$C_D = \frac{D}{\left(\frac{1}{2}\rho U_\infty^2\right)A} = \frac{\frac{2}{15}\rho b U_\infty^2 \sqrt{\frac{30}{Re_L}}\,L}{\left(\frac{1}{2}\rho U_\infty^2\right)(bL)} \approx \frac{1.461}{\sqrt{Re_x}}$$

$$C_F = \frac{\tau_w}{\frac{1}{2}\rho U_\infty^2} = \frac{\frac{2\mu U_\infty}{\sqrt{\frac{30}{Re_x}}\,x}}{\frac{1}{2}\rho U_\infty^2} \approx \frac{0.730}{\sqrt{Re_x}}$$

Note that we can calculate the drag force from

$$D = \int \tau_w(x)\,dA = b\int \tau_w(x)\,dx$$



$$= \frac{1}{2} b \rho u_\infty^2 \int_0^L C_F(x)\,dx$$

Summary of our results for $u(\eta) = u_\infty(2\eta - \eta^2)$

$$\frac{\delta}{x} \approx \frac{5.477}{\sqrt{Re_x}} \qquad \frac{\delta^*}{x} \approx \frac{1.72}{\sqrt{Re_x}} \qquad \frac{\theta}{x} \approx \frac{0.730}{\sqrt{Re_x}}$$

$$C_D = 2C_F \approx \frac{1.461}{\sqrt{Re_x}} \qquad H = \frac{\delta^*}{\theta} \approx 2.501$$

shape factor

Later, we will see that more accurate results can be obtained by Blasius theory. Let us see the resulting formulas.

Results from Blasius theory:

$$\frac{\delta}{x} \approx \frac{5.29}{\sqrt{Re_x}} \qquad \frac{\delta^*}{x} \approx \frac{1.72}{\sqrt{Re_x}} \qquad \frac{\theta}{x} \approx \frac{0.664}{\sqrt{Re_x}}$$

$$C_D = 2C_F \approx \frac{1.328}{\sqrt{Re_x}} \qquad H = \frac{\delta^*}{\theta} \approx 2.590$$

As we see, the approximate velocity profile works pretty well but slightly overestimates δ, δ^*, θ, C_D & C_F.

Note that these eqn's are valid for laminar boundary layers. Always remember to check

$$Re_L = \frac{\rho U_\infty L}{\mu} = \frac{U_\infty L}{\nu}$$

$Re < 2500$: thick BL $\quad \frac{\delta}{x} \approx 0.1$

$2500 < Re < 3 \times 10^6$: laminar BL

$3 \times 10^6 < Re < 5 \times 10^6$: turbulent BL

Problem: A sharp flat plate with $L = 1m$ and $b = 3m$ is immersed parallel to a stream of velocity $2m/s$

Find $\delta, \delta^*, \theta, D$ for

a) air $\rho = 1.23 \, kg/m^3$ $\quad \nu = 1.46 \times 10^5 \, m^2/s$

b) water $\rho = 1000 \, kg/m^3$ $\quad \nu = 1.02 \times 10^{-6} \, m^2/s$

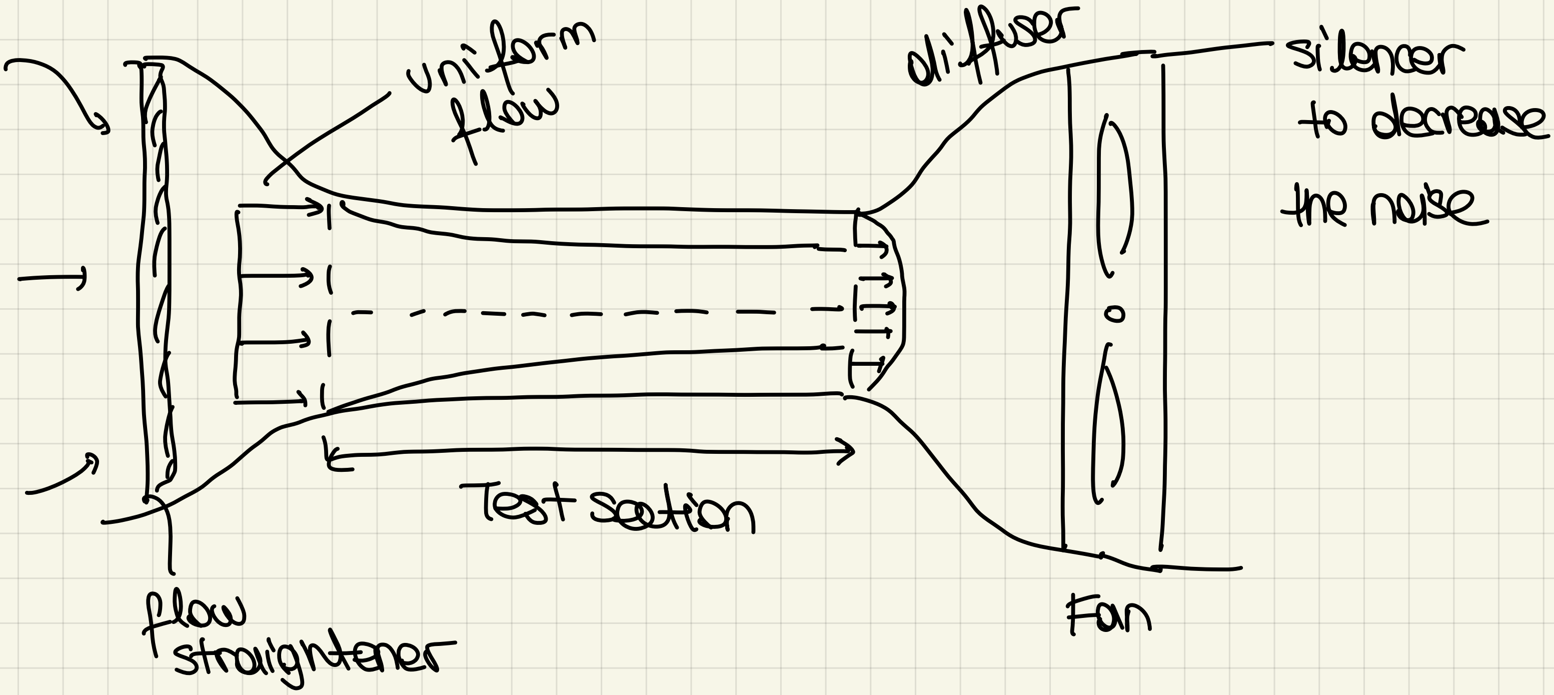
	Air	water
C_D	0.00359	0.000949
D	0.0265 N	5.7 N
δ	13.5 mm	3.57 mm
δ^*	4.65 mm	1.23 mm
θ	1.79 mm	0.48 mm
Re	0.137×10^6	1.96×10^6

For water, D is much larger but C_D, δ, δ^* and θ take smaller values.

Problem: A small low-speed wind tunnel is being designed. The test section is 30cm in diameter and 30 cm in length. The design has to be optimized for an airspeed of 4m/s.

a) Explain why there will be an acceleration of the centerline speed due to BL developed in the test section.

b) Estimate the increase in the centerline speed.



We assume 1) Flow is steady & incompressible $\rho = \rho_\infty$

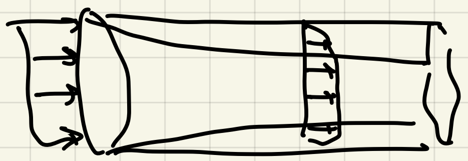
2) The walls are smooth. We will not look at the roughness effect but you should know that they are important.

3) Disturbances & vibrations are kept at minimum.

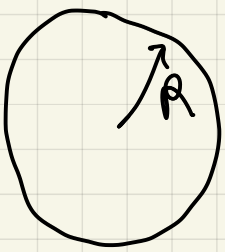
4) BL is laminar. (This should be checked)

$$Re_L = \frac{U_\infty L}{\nu} = \frac{4 \times 0.3}{1.507 \times 10^{-3}} = 7.96 \times 10^4 < 3 \times 10^6$$

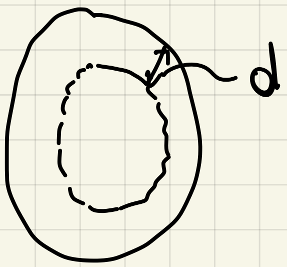
→ laminar



Streamlines will be shifted and the flow is effectively through a smaller area.



entrance



end of the test section

Cons. of mass: $\cancel{\rho} V_1 A_1 = \cancel{\rho} V_2 A_2$

$$\frac{V_2}{V_1} = \frac{A_1}{A_2} = \frac{\pi R^2}{\pi (R-d)^2}$$

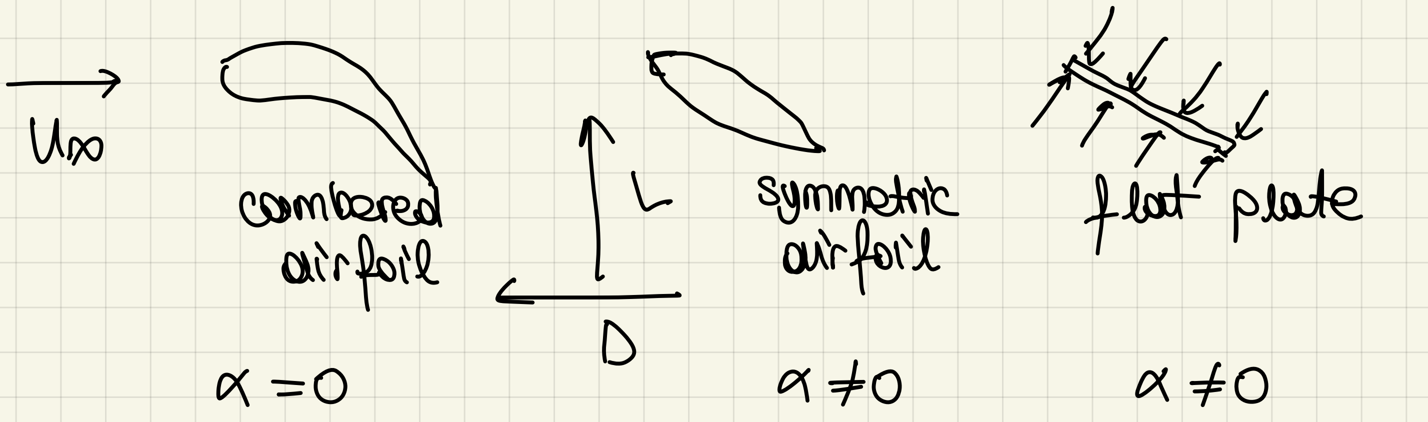
Hw: Find $\frac{V_2}{V_1}$ for

$d = \delta$, $d = \delta + \delta^*$, $d = \delta^*$

$V_2 \approx 1.02\dots \times V_1$

We find the real velocity dist. in Fluid Mech 2.

Let us consider a couple of scenarios to have a better understanding of the drag force.



⇒ Different pressure dist, and therefore pressure forces, on the lifting surfaces.

$$\vec{F}_p = -\int p\,\vec{n}\,dA$$

Lift force can be calculated fairly accurately by using inviscid theory along with the Kutta cond.

Kutta cond: It enforces numerically that the flow leaves the TE smoothly.

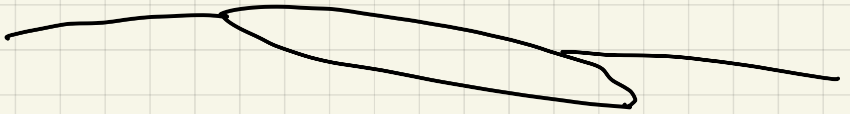
Navier-Stokes eqn's: $\Sigma\vec{F} = m\vec{a}$ applied to fluids

$$\rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \rho\frac{du}{dt} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

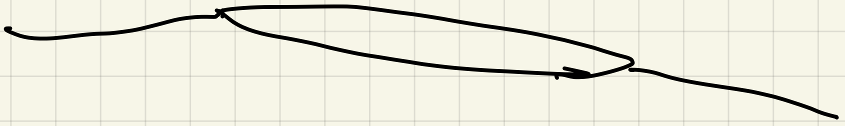
one for y-direction + one for z-direction

Inviscid theory: neglect viscosity effects $\mu \to 0$

$$\text{Navier-Stokes eqn's} \xrightarrow[\mu \to 0]{} \text{Euler's eqn's}$$



: zero lift



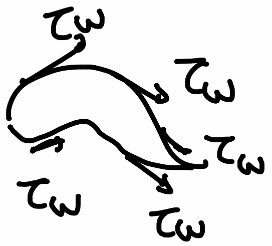
: Lift flow with Kutta cond.

However, if you calculate the drag force using the same theory, it results in zero drag.
This is what is commonly known as d'Alembert's paradox.

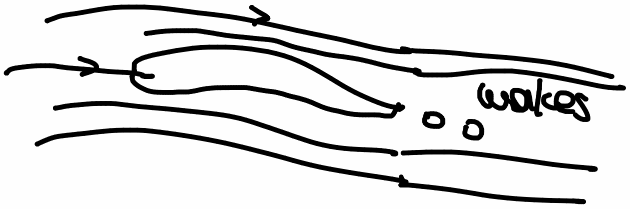
When we use viscid theory, the paradox is gone. Therefore, we can say that viscosity is solely responsible for aerodynamic drag on a lifting surface.

Drag: i) Skin friction drag: due to the shear stress acting on the surface

ii) Pressure drag: also called the form drag due to flow seperation



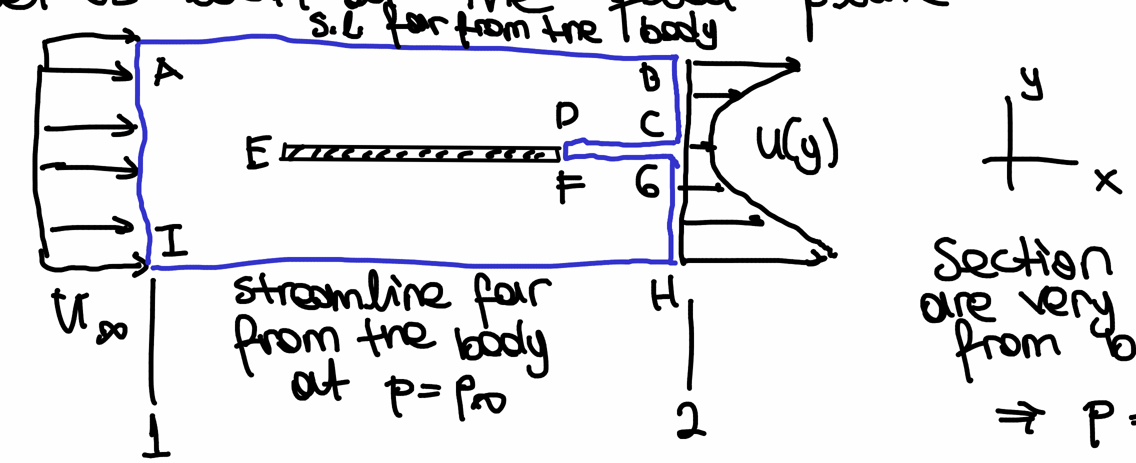
skin friction drag



pressure / form drag

We can use the momentum eqn. to evaluate the drag.

Let us look at the flat plate.



Section 1 & 2 are very far away from body

$\Rightarrow P = P_\infty$

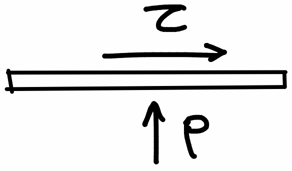
Control volume: ABCDEFGHIA

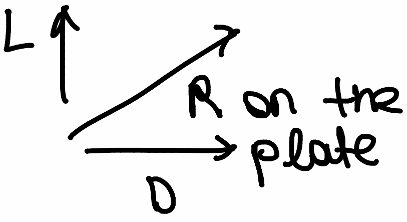
We will look at the surface forces to calculate the drag force:

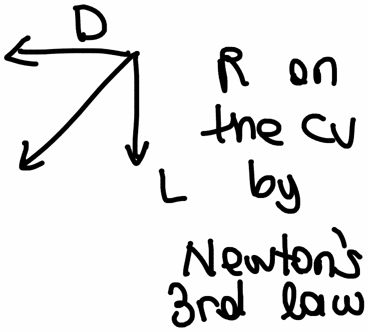
a) Pressure on ABHI : $-\int p\vec{n}\,dA$

b) surface forces on DEF due to the presence of the idea.

i) surface shear stress is neglected on AB & HI
ii) cuts CD & GF are adjacent to each other such that both shear & pressure forces cancel each other.
iii) surface force on DEF is total reaction







momentum eqn. in the x-dir.

$\Sigma F_x = -D$ since the pressure forces in the x-dir. cancel each other.

$-D = \int_{CS} \rho u (\vec{V} \cdot \vec{n}) dA$ for a steady flow.

$\searrow$ net momentum flux in the x-direction $\boxed{dA = b\,dy}$

$-D = -\rho b \int U_\infty^2 dy + \rho b \int u^2 dy$

$\Rightarrow D = \rho b \int (U_\infty^2 - u^2) dy$

continuity: $0 = \int_{CS} \rho (\vec{V} \cdot \vec{n}) dA$

$0 = -\rho b \int U_\infty dy + \rho b \int u dy$

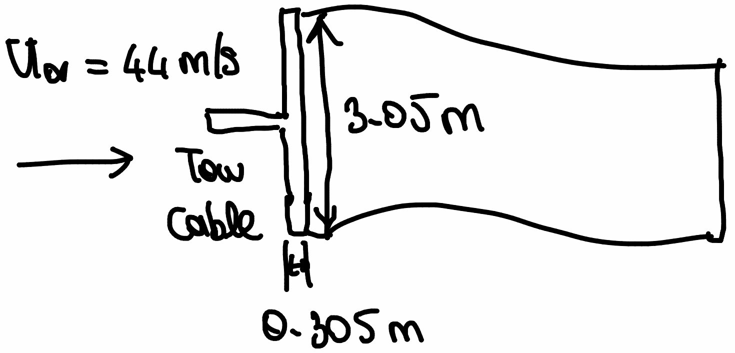
$\Rightarrow \int U_\infty dy = \int u dy \rightarrow \int U_\infty^2 dy = \int U_\infty u dy$

Use this in our previous result for D:

$D = \rho b \int (U_\infty u - u^2) dy = \rho b U_\infty^2 \int \left[\frac{u}{U_\infty} - \left(\frac{u}{U_\infty}\right)^2 \right] dy$

$$\boxed{D = \rho b U_\infty^2 \int \frac{u}{U_\infty} \left(1 - \frac{u}{U_\infty}\right) dy}$$

Problem: A thin plate 3.05m tall and 0.305m wide form the leading edge of a banner towed by an aircraft at 44 m/s. ($\nu = 1.52 \times 10^{-5} m^2/s$)



a) How far from the LE of the plate does the laminar portion of the BL extend.

b) $\delta = ?$ at the downstream of the laminar BL

c) Drag force on the laminar part.

a) $Re_x = \frac{U_\infty x}{\nu}$ The transition to turbulent portion starts at $Re_x = 5 \times 10^6$

$$x = \frac{Re_x \nu}{U_\infty} = \frac{5 \times 10^6 \times 1.52 \times 10^{-5}}{44} = 0.17m$$

b) $\frac{\delta}{x} \approx \frac{5.29}{\sqrt{Re_x}}$ Hw: Find δ.

c) $C_D \approx \frac{1.328}{\sqrt{Re_x}} = \frac{D}{(\frac{1}{2}\rho U_\infty^2)(bx)}$ $x = 0.17m$ $b = 3.05$ m

Hw! Complete the calculation & find the drag force.

With this, we are done with the CV analysis. Next week: we will derive the diff eqn.s for the laminar BL, Prandtl's eqn.s.

Navier-Stokes eqn.s $\xrightarrow{\text{simplify}}$ Prandtl's eqn.s

hard to solve

relatively easy to solve

Derivation of Prandtl's Laminar BL eqn's:

So far, we have only performed a CV analysis by using Reynold's transport theorem and assumed $u(y)$ is given.

Now, we will derive the diff. eqn's for a laminar BL and see how we can find the velocity profile, and parameters charactering a BL ($\delta, \delta^*, \theta, C_F, C_D$) numerically.

This is a must when we deal with geometries that are more complicated compared to a flat plate, which are commonly needed in engineering applications.

Q: Why don't we directly use the Navier-Stokes equations?

This is a common practice in fluid mechanics. By making use of the physical characteristics of the systems that we study, we can obtain eqn's that are easier to solve.

Navier-Stokes eqn.s: $\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \vec{\nabla} p + \mu \nabla^2 \vec{V}$

$$\underbrace{\vec{\nabla}}_{\text{nabla}} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \qquad \underbrace{\nabla^2}_{\text{Laplacian}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\rho \frac{d\vec{V}}{dt} = \rho \vec{g} - \vec{\nabla} p + \mu \nabla^2 \vec{V} \qquad \vec{\nabla} p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+(\vec{V}\cdot\vec{\nabla})v\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\vec{\nabla}^2 v$$

$$\rho\left(\frac{\partial w}{\partial t}+(\vec{V}\cdot\vec{\nabla})w\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\vec{\nabla}^2 w$$

Non-dimensionalize the eqn's using the following:
velocity: V_∞ length: l pressure: ρV_∞^2 time: $\frac{l}{V_\infty}$

$$u^*=\frac{u}{V_\infty}\qquad v^*=\frac{v}{V_\infty}\qquad w^*=\frac{w}{V_\infty}\qquad p^*=\frac{p}{\rho V_\infty^2}$$

$$t^*=\frac{t}{l/V_\infty}=\frac{tV_\infty}{l}$$

Check comp.s by assuming 2d flow & $\vec{g}=0$.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

Divide by V_∞:

$$\frac{\partial u^*}{\partial t}+u^*\frac{\partial u}{\partial x}+v^*\frac{\partial u}{\partial y}=-\frac{1}{\rho V_\infty}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u^*}{\partial x^2}+\frac{\partial^2 u^*}{\partial y^2}\right)$$

Multiply by $\frac{l}{V_\infty}$:

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}=-\frac{l}{\rho V_\infty^2}\frac{\partial p}{\partial x}+\frac{\mu l}{\rho V_\infty}\left(\frac{\partial^2 u^*}{\partial x^2}+\frac{\partial^2 u^*}{\partial y^2}\right)$$

$$\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\mu \ell^2}{\rho V_\infty \ell}\left(\frac{\partial u^*}{\partial x^2} + \frac{\partial^2 u^*}{\partial y^2}\right)$$

$$\boxed{\frac{\partial u^*}{\partial t^*} + u^*\frac{\partial u^*}{\partial x^*} + v^*\frac{\partial u^*}{\partial y^*} = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial^2 u^*}{\partial y^{*2}}\right)}$$ ①

momentum eqn. in the x-direction.

$$\boxed{\frac{\partial v^*}{\partial t^*} + u^*\frac{\partial v^*}{\partial x^*} + v^*\frac{\partial v^*}{\partial y^*} = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}} + \frac{\partial^2 v^*}{\partial y^{*2}}\right)}$$ ②

momentum eqn. in the y-direction

Let's look at the cont. eqn.

$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{V}) = 0$ Incomp. flow: ρ = constant

$\vec{\nabla}\cdot\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ multiply by $\frac{\ell}{V_\infty}$

$$\boxed{\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0}$$ ③

continuity eqn. in dimensionless form.

Remember: $\frac{\delta}{\ell} \sim \frac{1}{\sqrt{Re_\ell}}$ ⇒ δ is inv. proportional to $\sqrt{Re_\ell}$

As $Re_\ell \to \infty$, $\delta \to 0$. For large Re, the viscous term with $\nabla^2\vec{V}$ becomes negligibly small.

This reduces to the inviscid case & ① and ② reduce to Euler's eqn.s.

That means if we solve NS eqn's at large Re, it is hard to take viscous effects into account.

This is a problem since we hope to solve these eqn's inside the BL where viscous effects dominate. We need to make sure that as $Re \to \infty$, the viscous terms do not dissapear. They should be comparable to other terms in the eqn's

$x^* = \frac{x}{l} \equiv O(1)$ $u^* = \frac{u}{V_\infty} \equiv O(1)$ since V_∞ is U_{max}.

$y^* = \frac{y}{l} \sim \frac{\delta}{l} = \delta^* \equiv O(\delta^*)$ we write $y \sim \delta$ since $y_{max} = \delta$

As $Re \to \infty$, $\delta \to 0$ and $\delta^* \to 0$. Let's look at the continuity eqn.

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

This term becomes problematic as $Re \to \infty$ unless $v^* \equiv O(\delta^*)$

We need to have $v^* = O(\delta^*)$ to avoid a degenerate continuity eqn.

Let us now write the order of magnitude of each term in the momentum eqn's:

$$\underset{\substack{\sim\\ O(1)}}{\frac{\partial u^*}{\partial t^*}} + \underset{\substack{\sim\\ O(1)}}{u^* \frac{\partial u^*}{\partial x^*}} + \underbrace{v^* \frac{\partial u^*}{\partial y^*}}_{\delta^* \frac{1}{\delta^*} = O(1)} = -\underset{\substack{\sim\\ O(1)}}{\frac{\partial p^*}{\partial x^*}} + \underset{\substack{\sim\\ O(\delta^{*2})}}{\frac{1}{Re}} \left(\underset{\substack{\sim\\ O(1)}}{\frac{\partial^2 u^*}{\partial x^{*2}}} + \underset{\substack{\sim\\ O\left(\frac{1}{\delta^{*2}}\right)}}{\frac{\partial^2 u^*}{\partial y^{*2}}} \right)$$

$t^* = O(1)$

To get rid of the $O\left(\frac{1}{\delta^{*2}}\right)$ term, we should have

$Re = O\left(\frac{1}{\delta^{*2}}\right)$

What about the momentum eqn. in the y-direction?

$$\underbrace{\frac{\partial v^*}{\partial t^*}}_{O(\delta^*)}+\underbrace{u^*\frac{\partial v^*}{\partial x^*}}_{O(\delta^*)}+\underbrace{v^*\frac{\partial v^*}{\partial y^*}}_{O(\delta^*)}=-\underbrace{\frac{\partial p^*}{\partial y^*}}_{?}+\underbrace{\frac{1}{Re}}_{O(\delta^{*2})}\left[\underbrace{\frac{\partial^2 v^*}{\partial x^{*2}}}_{O(\delta^*)}+\underbrace{\frac{\partial^2 v^*}{\partial y^{*2}}}_{O(1/\delta^*)}\right]$$

Additionally, we will also use the following variables:

$$\bar{y}\equiv y^*\sqrt{Re}=O(1)$$
$$\bar{v}\equiv v^*\sqrt{Re}=O(1)$$

Let's see the continuity eqn:

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial(v^*\sqrt{Re})}{\partial(y^*\sqrt{Re})}=\frac{\partial u^*}{\partial x^*}+\frac{\partial\bar{v}}{\partial\bar{y}}=0$$

This is not degenerate & all the terms are $O(1)$.

Momentum eqn.s for high Re ($Re\to\infty$, $\delta\to 0$, $\delta^*\to 0$)

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+\bar{v}\frac{\partial u^*}{\partial\bar{y}}=-\frac{\partial p^*}{\partial x^*}+\frac{\partial^2 u^*}{\partial\bar{y}^2}\quad\text{--- x-comp.}$$

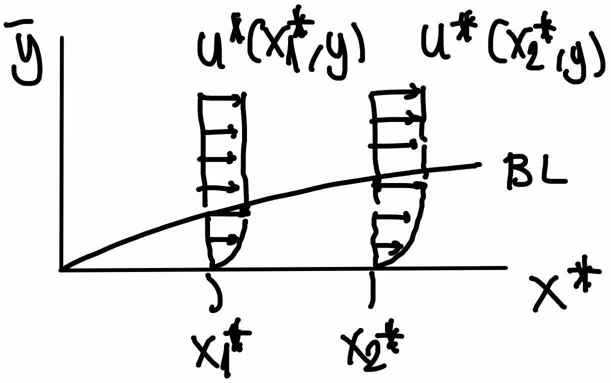
$$\frac{\partial p^*}{\partial\bar{y}}=0\quad\text{--- y-comp.}$$

As a result, we obtain

$$\boxed{\begin{array}{l}\dfrac{\partial u^*}{\partial t^*}+u^*\dfrac{\partial u^*}{\partial x^*}+\bar{v}\dfrac{\partial u^*}{\partial\bar{y}}=-\dfrac{\partial p^*}{\partial x^*}+\overset{\text{viscous term}}{\dfrac{\partial^2 u^*}{\partial\bar{y}^2}}\quad\text{---- (i)}\\[2ex]\dfrac{\partial p^*}{\partial\bar{y}}=0\quad\text{---- (ii)}\qquad\dfrac{\partial u^*}{\partial x^*}+\dfrac{\partial\bar{v}}{\partial\bar{y}}=0\quad\text{---- (iii)}\end{array}}$$

We have not reached the final form of the BL eqn.s yet & further simplifications are possible: Pressure is independent of $\bar{y}$.

$$p^*=p^*(x^*,t^*)$$



At the edge of the BL

$$\frac{\partial u^*}{\partial \bar{y}} = 0 \qquad \frac{\partial^2 u^*}{\partial \bar{y}^2} = 0$$

Check (i) at the edge of the BL:

$$-\frac{\partial p}{\partial x^*} = \frac{\partial u_\infty^*}{\partial t^*} + u_\infty^* \frac{\partial u_\infty^*}{\partial x^*}$$

: der. of pressure in terms of u_∞^*

We will use this in (i) again:

$$\frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + \bar{v} \frac{\partial u^*}{\partial \bar{y}} = \frac{\partial u_\infty^*}{\partial t^*} + u_\infty^* \frac{\partial u_\infty^*}{\partial x^*} + \frac{\partial^2 u^*}{\partial \bar{y}^2} \quad \text{----} ①$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad \text{----} ②$$

This reduces to # of eqn.s to 2.

Boundary Conds: when $\bar{y} = 0$, $u^* = 0$ and $v^* = 0$
$\bar{y} \to \infty$, $u^* = u_\infty^*(x^*, t^*)$

We can simplify these eqn.s even further by considering a steady-state case: $\partial/\partial t^* \to 0$

In such a case, $-\frac{\partial p^*}{\partial x^*} = -\frac{dp^*}{dx^*} = u_\infty^* \frac{\partial u_\infty^*}{\partial x^*} = u_\infty^* \frac{du_\infty^*}{dx^*}$

$$u^* \frac{\partial u^*}{\partial x^*} + \bar{v} \frac{\partial u^*}{\partial \bar{y}} = u_\infty^* \frac{du_\infty^*}{dx} + \frac{\partial^2 u^*}{\partial \bar{y}^2} \quad \text{---} ③$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \quad \text{----} ④$$

These are the Prandtl's BL eqn.s and

they are indep. of Re! Therefore, we need to solve it only once.

Boundary cond.s: At $\bar{y}=0$, $u^*=0$, $\bar{v}=0$
$\bar{y}\to\infty$, $u^*=u^*_\infty(x^*)$
(no t-dep.)

(i)(ii)(iii): elliptic PDE system
③ ④: parabolic PDE system.

Details are beyond the scope of this course but you should know that the numerical nature of these eqn.s are different & the latter is much easier to solve.

Once we solve these eqn.s, we can easily determine $\delta, \delta^*, \theta, C_D, C_F, D, \tau_w \ldots$

For example, $C_F = \dfrac{\tau_w}{\frac{1}{2}\rho u_\infty^2} = \dfrac{\mu \frac{du}{dy}\big|_{y=0}}{\frac{1}{2}\rho u_\infty^2} = \dfrac{2}{\sqrt{Re}}\dfrac{\partial u^*}{\partial \bar{y}}\Big|_{\bar{y}=0}$

HW: Show it!

As we see, $C_F \to 0$ as $Re\to\infty$.

We have used both knowledge of physics & numerical behaviour of eqn.s to come up with a simpler set of eqn.s.

A lot of these eqn.s can get numerically very involved. In order to handle them, we need the theory of linear algebra.

Exercise: $A=\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}$ $\quad v=\begin{bmatrix}4\\ -4\end{bmatrix}$

$$Av = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}4\\ -4\end{bmatrix} = \begin{bmatrix}8\\ -8\end{bmatrix} = 2\begin{bmatrix}4\\ -4\end{bmatrix} = 2v$$

$Av = 2v \Rightarrow \begin{bmatrix}4\\ -4\end{bmatrix}$ is an eigenvector of the matrix A with eigenvalue 2.

$$A\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix} = \begin{bmatrix}1\\ 3\end{bmatrix} \neq \lambda\begin{bmatrix}0\\ 1\end{bmatrix}$$

$\Rightarrow$ It's not an eigenvector.

To study eigenvalues & eigenvectors systematically we look at the eqn:

$$Av = \lambda v = \lambda \mathbb{1} v \Rightarrow (A - \lambda\mathbb{1})v = 0$$

$\mathbb{1}$: identity matrix

If $(A-\lambda\mathbb{1})$ has an inverse, then $v=0$ is the only soln. Therefore, we should check when it does not have an inverse, i.e., $\det(A-\lambda\mathbb{1}) = 0$.

$$\det(A-\lambda\mathbb{1}) = \begin{vmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix} = \lambda^2 - 6\lambda + 9 - 1$$
$$= \lambda^2 - 6\lambda + 8$$
$$= (\lambda-2)(\lambda-4) = 0$$

$\lambda_1 = 2$ & $\lambda_2 = 4$ are eigenvalues.

For $\lambda_1 = 2$, we have $\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix} = \begin{pmatrix}0\\ 0\end{pmatrix}$

$v_1 + v_2 = 0$ Choose $v_1 = 1$. Then $v_2 = -1$.

Therefore, $v^{(1)} = \begin{pmatrix}1\\ -1\end{pmatrix}$ is the eigenvector

Corresponding to $\lambda_1 = 2$.

For $\lambda_2 = 4$, we have $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-v_1 + v_2 = 0$. Choose $v_1 = 1$ gives $v_2 = 1$

The second eigenvector $v^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

2nd Mid-term Exam: Take home exam

Start: December 10 at 12:00 End: December 18 at 12:00

$u(y) = u_\infty(2y - y^2) \longrightarrow \delta, \delta^*, \theta, D, \tau_w, C_D, C_F$

No final exam but Final Project

We started with NS eqn's & derived the governing eqn's of a laminar BL. Now, we will try to get a solution for the flat plate. This was first done by Blasius in 1908.

Similarity Soln. to the BL eqn's:

We apply Prandtl's BL eqn's to a flat plate by assuming U_∞ = const. (not a function of x)

BL eqn's: $$\underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}}_{\text{conv. acceleration terms}} = \underbrace{\nu\frac{\partial^2 u}{\partial x^2}}_{\text{effect of the viscous force}} \Rightarrow \text{momentum eqn. in the x direction}$$

It also contains information about the y-direction through the pressure distribution (Check last week's derivation & you will see)

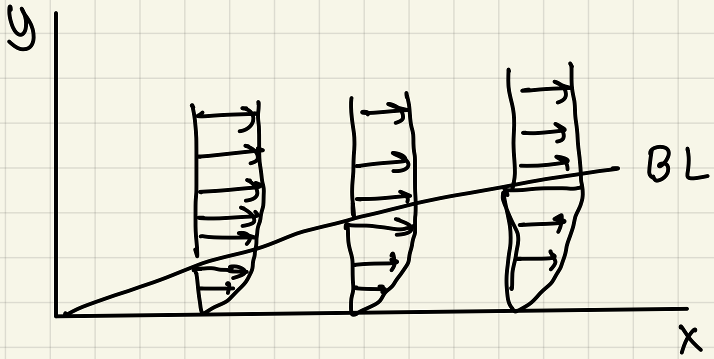
continuity eqn: $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Note that we recovered the original variables since we are done with the order-of-magnitude analysis.

BC's: At $y=0$ $\quad u=0 \quad v=0$

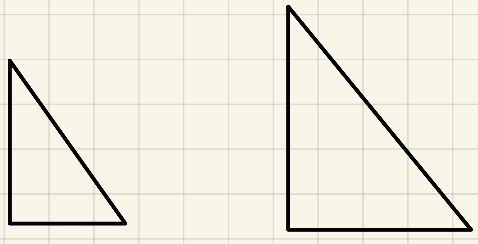
$y \to \infty \quad u = U_\infty$

Q: what is a similarity solution?



Geometrically, velocity profiles are "similar"

As a result, velocity profiles can be mapped to each other at different values of x by using a suitable scale factor for y & u.



For similar triangles, the scale factor is a number.

But, here we need to use a position dependent scale factor.

To impose them, we also need the stream function ψ, which is very useful in solving 2d incompressible flow problems.

Let's consider the continuity eqn:

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ If $u = \frac{\partial \psi}{\partial y}$ and $v = -\frac{\partial \psi}{\partial x}$,

the continuity eqn. is automatically satisfied. This reduces the number of unknowns to 1 & we can solve the momentum eqn. for the stream function ψ.

Important property: ψ = constant along a streamline.

Therefore, plotting ψ = constant curves gives us streamlines. Let's prove it:

$\frac{dx}{u} = \frac{dy}{v}$ for a streamline in 2d

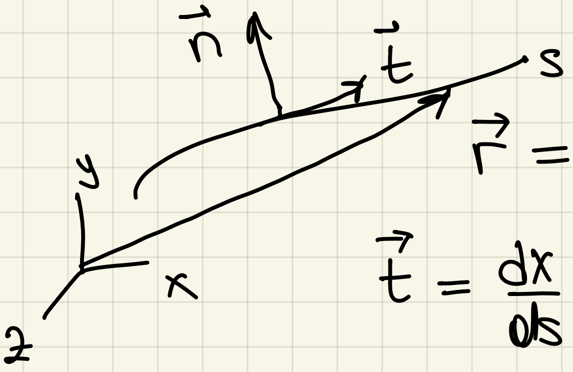
$$0 = u\,dy - v\,dx = \frac{\partial \psi}{\partial y} dy - \left(-\frac{\partial \psi}{\partial x}\right) dx$$

$$= \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$

That means ψ = const. along a streamline. This is the geometrical interpretation of the stream function.

There is also an important physical interp. that relates ψ to the volume flow rate Q.

$$dQ = \vec{V}\cdot\vec{n}\,dA$$



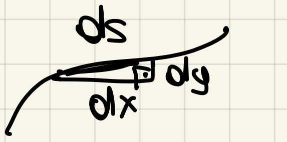
$$\vec{r} = (x(s), y(s), z(s))$$

$$\vec{t} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$$

Since $\vec{t}\cdot\vec{n} = 0$, we can deduce $\vec{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$

$$\vec{t}\cdot\vec{n} = \frac{dx}{ds}\frac{dy}{ds} - \frac{dy}{ds}\frac{dx}{ds} = 0$$

Remember, $\vec{n}$ is a unit normal vector.



$$ds^2 = dx^2 + dy^2$$

$$|\vec{n}| = \sqrt{\left(\frac{dy}{ds}\right)^2 + \left(\frac{dx}{ds}\right)^2} = \sqrt{\frac{dy^2+dx^2}{ds^2}}$$

$= 1$ as it should be.

$$dQ = \vec{V}\cdot\vec{n}\,dA = \left(\hat{i}\frac{\partial\psi}{\partial y} - \hat{j}\frac{\partial\psi}{\partial x}\right)\cdot\left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}\right)(ds\cdot 1)$$

volume flow rate per unit depth.

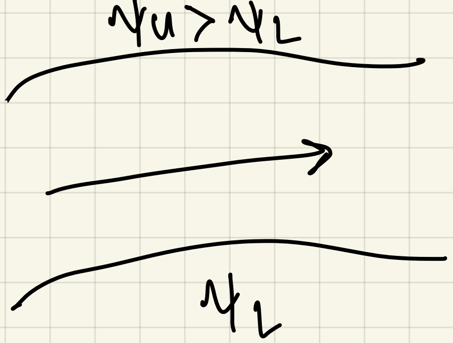
$$= \left(\frac{\partial\psi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\psi}{\partial x}\frac{\partial x}{\partial s}\right)ds = \frac{d\psi}{ds}ds = d\psi$$

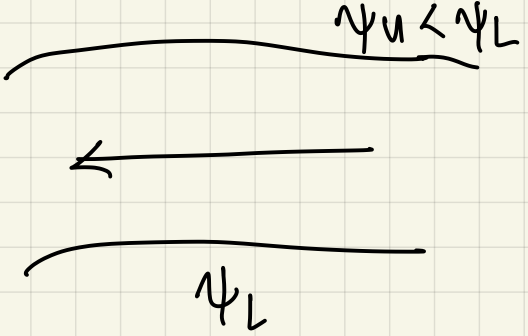
Therefore, the change in ψ across the element is numerically equal to the volume flow rate across the element

$$Q_{1\to 2} = \int_1^2 \vec{V}\cdot\vec{n}\,dA = \int_1^2 d\psi = \psi_2 - \psi_1$$

volume flow rate between two streamlines

The sign convention is as follows





mass flow rate (for a incompressible flow):

$$m_{1\to 2} = \rho\, Q_{1\to 2}$$

Another important object is the vorticity, which is defined as

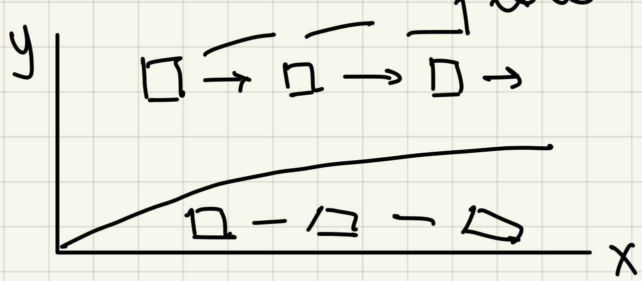
$$\vec{\xi} = \text{curl}\,\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

$$\overset{\text{2d flow}}{=} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

It characterizes the viscous structure of the flow:

$\vec{\xi} = 0 \Rightarrow$ irrotational flow $\Rightarrow$ inviscid flow

$\vec{\xi} \neq 0 \Rightarrow$ rotational flow $\Rightarrow$ viscid flow



The similarity transformations that we will use are as follows:

$$\underbrace{\eta}_{\text{similarity variable}} = y\sqrt{\frac{U_\infty}{2\nu x}} \qquad \psi(x,y) = \sqrt{2\nu U_\infty x}\, f(\eta)$$

$$\frac{\partial \eta}{\partial x} = y\sqrt{\frac{U_\infty}{2\nu}}\left(-\frac{1}{2x\sqrt{x}}\right) = -\frac{\eta}{2x}\ ; \quad \frac{\partial \eta}{\partial y} = \sqrt{\frac{U_\infty}{2\nu x}}$$

$$u = \frac{\partial \psi}{\partial y} = \sqrt{2\nu U_\infty x}\,\frac{\partial f}{\partial y} = \sqrt{2\nu U_\infty x}\,\frac{\partial \eta}{\partial y}\frac{df}{d\eta}$$

$$= \sqrt{2\nu x U_\infty}\sqrt{\frac{U_\infty}{2\nu x}}\, f' = U_\infty f'$$

$$v = -\frac{\partial \psi}{\partial x} = -\sqrt{2\nu U_\infty}\,\frac{1}{2x\sqrt{x}}\, f + \sqrt{2\nu U_\infty x}\,\underbrace{\frac{\partial f}{\partial x}}_{\frac{df}{d\eta}\frac{\partial \eta}{\partial x}}$$

$$\vdots$$

$$= \sqrt{\frac{\nu U_\infty}{2x}}\,(\eta f' - f)$$

$$\vdots$$

Insert everything into the momentum eqn:

$$\boxed{f''' + f f'' = 0} \quad \text{Blasius eqn.}$$

BC's: $y=0$ $u=0$ $v=0$

$\eta=0$ $f'(0)=0$ $f(0)=0$

$y\to\infty$ $u=U_\infty$

$\eta\to\infty$ $f'(\infty)=1$

Summary: $f''' + f f'' = 0$: Blasius eqn.

B.C's : $\eta=0$ $f(0)=0$ $f'(0)=0$

$\eta\to\infty$ $f'(\infty)=1$

Once we determine f, we can find ψ, u, v streamlines, Q, $\dot{m}$, δ, δ^*, θ, ...

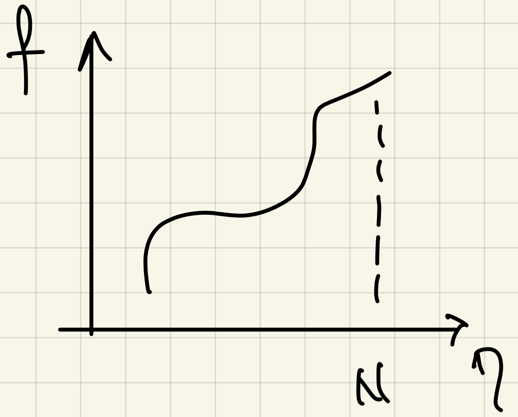
There is a practical problem:

1) No analytical solution.

2) For a numerical soln, we need the initial conditions: $f(0)=0$, $f'(0)=0$, $f''(0)=?$

We don't know $f''(0)$ but instead we know the BC at $\eta\to\infty$, i.e., $f'(\infty)=1$

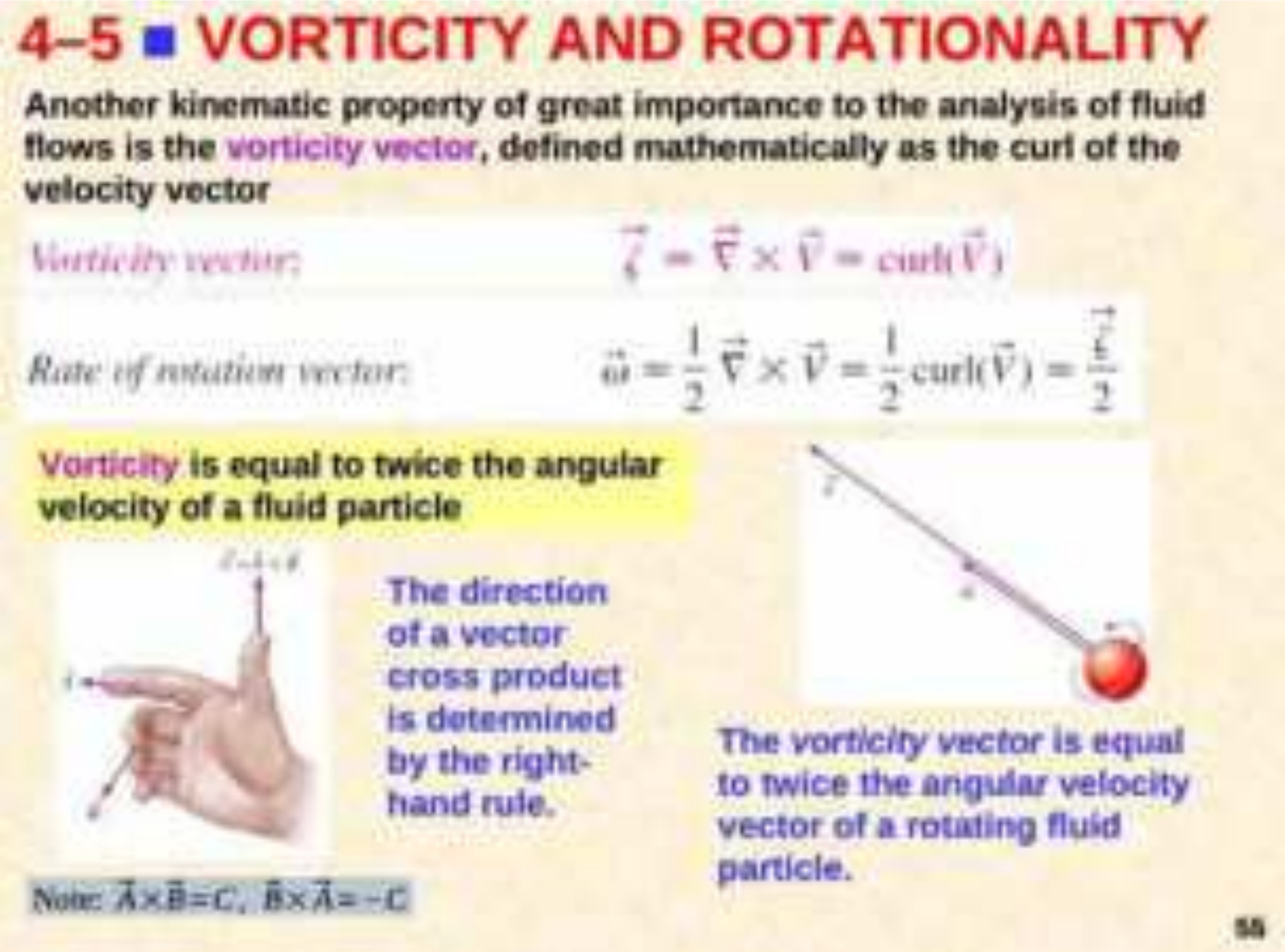
If we also know $f''(0)=A=\text{const}$, we would be able to find f numerically & plot it



We use the so-called shooting method

η very large, say 10^5

We choose different values of A to find f numerically until we get a solution satisfying $f'(N) \approx 1$



4–5 ■ VORTICITY AND ROTATIONALITY

Another kinematic property of great importance to the analysis of fluid flows is the vorticity vector, defined mathematically as the curl of the velocity vector

Vorticity vector:

$$\vec{\zeta} = \vec{\nabla} \times \vec{V} = \text{curl}(\vec{V})$$

Rate of rotation vector:

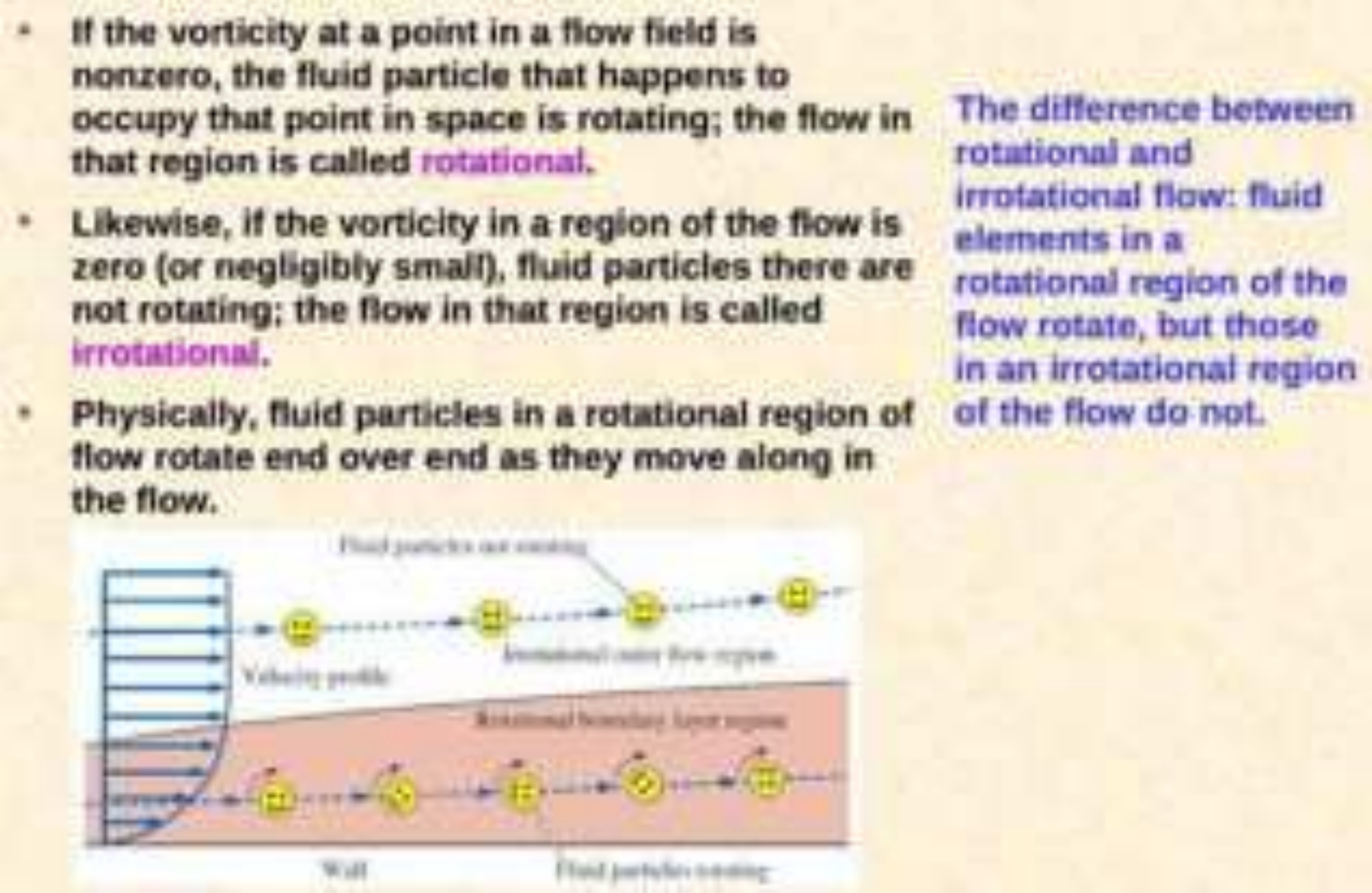
$$\vec{\omega} = \frac{1}{2}\vec{\nabla} \times \vec{V} = \frac{1}{2}\text{curl}(\vec{V}) = \frac{\vec{\zeta}}{2}$$

Vorticity is equal to twice the angular velocity of a fluid particle

The direction of a vector cross product is determined by the right-hand rule.

The *vorticity vector* is equal to twice the angular velocity vector of a rotating fluid particle.

Note: $\vec{A} \times \vec{B} = C$, $\vec{B} \times \vec{A} = -C$


Fluid particles not rotating
Irrotational outer flow region
Velocity profile
Rotational boundary layer region
Wall
Fluid particles rotating

- If the vorticity at a point in a flow field is nonzero, the fluid particle that happens to occupy that point in space is rotating; the flow in that region is called rotational.
- Likewise, if the vorticity in a region of the flow is zero (or negligibly small), fluid particles there are not rotating; the flow in that region is called irrotational.
- Physically, fluid particles in a rotational region of flow rotate end over end as they move along in the flow.

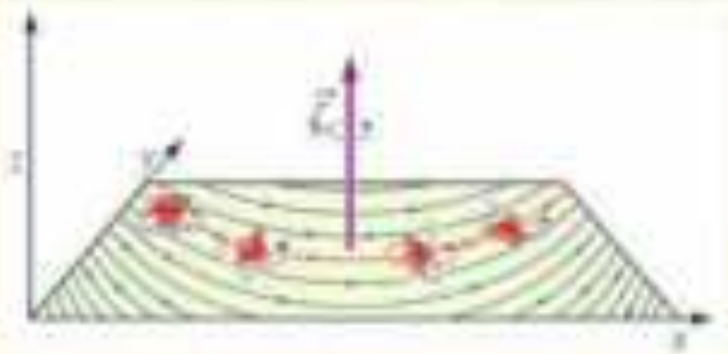
The difference between rotational and irrotational flow: fluid elements in a rotational region of the flow rotate, but those in an irrotational region of the flow do not.

Vorticity vector in Cartesian coordinates:

$$\vec{\zeta}=\vec{\nabla}\times\vec{V}, \qquad \vec{\zeta} = \left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\vec{k}$$

Two-dimensional flow in Cartesian coordinates:

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\vec{k}$$



For a two-dimensional flow in the *xy*-plane, the vorticity vector always points in the *z*-direction. In this illustration, the flag-shaped fluid particle rotates in the counterclockwise direction as it moves in the *xy*-plane; its vorticity points in the positive *z*-direction as shown.